CMR COLLEGE OF ENGINEERING & TECHNOLOGY
(UGC - AUTONOMOUS)
ODEs AND MULTIVARIABLE CALCULUS
(Common to all branches)

CODE: A30005

L	T	P	C
3	1	0	4

UNIT-I

First Order ODE: Exact, Linear and Bernoulli's differential equations, Applications, Newton's law of cooling, Law of natural growth and decay.

Equations not of first degree: Equations solvable for p, Equations solvable for y, Equations solvable for x and Clairaut's type.

UNIT -II

Ordinary Differential Equations of Higher Order: Second and higher order linear differential equations with constant coefficients, Non-Homogeneous terms of the type e^{ax}, $\sin ax$, $\cos ax$, polynomials in x, $e^{ax}V(x)$ and $x\,V(x)$, Method of variation of parameters, Equations reducible to linear ODE with constant coefficients, Legendre's equation, Cauchy-Euler equation.

UNIT -III

Multivariable Calculus (Integration): Evaluation of Double Integrals (Cartesian and polar coordinates), Change of order of integration (only Cartesian form), Evaluation of Triple Integrals, Change of variables (Cartesian to polar) for double and (Cartesian to Spherical and Cylindrical polar coordinates) for triple integrals, **Applications:** Areas (by double integrals) and volumes (by double integrals and triple integrals).

UNIT -IV

Vector Differentiation: Vector point functions and scalar point functions, Gradient, Divergence and Curl. Directional derivatives, Tangent plane and normal line, Vector Identities, Scalar potential functions, Solenoidal and Irrotational vectors.

UNIT -V

Vector Integration: Line, Surface and volume Integrals. Theorems of Green's, Gauss and Stoke's(without proofs) and their applications.

TEXT BOOKS :

1. Higher Engineering Mathematics, (36th Edition), B.S. Grewal, Khanna Publishers, 2010
2. Advanced Engineering Mathematics, (9th Edition), Erwin kreyszig, John Wiley & Sons,2006.

REFERENCE BOOKS:

1. Advanced Engineering Mathematics(3rd edition) by R.K. Jain & S.R.K. Iyengar, Narosa Publishing House, Delhi.
2. Differential Equations with Applications & Historical Notes(2nd Edi) by George F Simmons, Tata Mc. graw Hill Publishing Co Ltd.
3. Advanced Engineering Mathematics(8th Edition) by Kreyszig, John Wiley & Sons Publishers
4. G.B. Thomas and R.L. Finney, Calculus and Analytic geometry(9th Edition), Pearson, Reprint, 2002
5. Mathematics for Engineering and Scientists(6th Edi), by. Alan Jeffrey, 2013, Chapman & Hall / CRC
6. Engineering Mathematics – I by T.K.V. Iyengar, B. Krishna Gandhi & Others, 2012 Yr. Edition S.Chand.
7. Differential Equations(3rd Ed), S. L. Ross Wiley India, 1984.

Ravish Singh
M-D-Raisingalia.

COURSE OUTCOMES:

On completion of the course students will be able to

1. Determine first order differential equations and obtain solutions.
2. Solve higher order linear differential equations using various methods.
3. Evaluate areas and volumes using multiple integrals .
4. Evaluate Gradient, Divergence, Curl and directional derivatives.
5. Evaluate integrals by converting line to surface integral and surface to volume integrals.

—: Differentiation and Integration Formulae :—

1. $d(x^n) = nx^{n-1}$
2. $d(e^{ax}) = ae^{ax}$
3. $d(\sin ax) = a\cos ax$
4. $d(\cos ax) = -a\sin ax$
5. $d(\log x) = 1/x$
6. $d(uv) = udv + vdu$
7. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$
8. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
9. $\int e^{ax} dx = \frac{e^{ax}}{a} + c$
10. $\int \cos ax\, dx = \frac{\sin ax}{a} + c$
11. $\int \sin ax\, dx = -\frac{\cos ax}{a} + c$
12. $\int 1/x\, dx = \log x + c$
13. $\int uv\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)dx$
14. $\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$
15. $\int a^x dx = \frac{a^x}{\log a} + c$
16. $\int \sin x\, dx = -\cos x + c.$
17. $\int \tan x\, dx = \log|\sec x| + c.$

(18). $\int \cot x\, dx = \log|\sin x| + c.$

(19). $\int \sec x\, dx = \log|\sec x + \tan x| + c$

(20). $\int \operatorname{cosec} x\, dx = \log|\csc x - \cot x| + c$

(21). $\int \sec^2 x\, dx = \tan x + c$

(22). $\int \csc^2 x\, dx = -\cot x + c$

(23). $\int 1/x^2+a^2\, dx = \frac{1}{a}\tan^{-1}(x/a) + c$

(24). $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

(25). $\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(26). $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}(x/a) + c$

(27). $\int 1/\sqrt{a^2+x^2}\, dx = \sinh^{-1}(x/a) + c$

(28). $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \cosh^{-1}(x/a) + c$

(29). $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c$

(30). $\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}(x/a) + c.$

(31). $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}(x/a) + c$

(32). $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

(33). $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

* ODEs AND MULTIVARIABLE CALCULUS * (1) (A30W5)

UNIT-I : First order ODE

UNIT-II : Ordinary Differential Eq^ns of higher order.

UNIT-III : Multivariable Calculus (Integration)

UNIT-IV : Vector Differentiation

UNIT-V : Vector Integration

Differential Eq^n :- (D.E).

* An eq^n containing dependent variables and Independent variables and differential coefficients of dependent variables i.e. w.r.to Independent variables is called "D.E."

Def :- An Eq^n involving derivatives of one (or) more dependent variables w.r.to one or more independent variables is called a "D.E."

* Derivative means rate measure.
* $\frac{dy}{dx}$ means rate at which the dependent variable 'y' is going to change an independent variable 'x'.

There are two types of D.Es. (i) ODE (ii) PDE.

(i). ODE :- If a D.E has only one Independent variable then it is called an "ODE" (or) If the derivatives in the Eq^n are ordinary then it is called "ODE". Eg:- $d^2y/dx^2 + 3dy/dx + 2y = 0$.

(ii) PDE :- If a D.E has two (or) more Independent variable (or) If the derivatives in the Eq^n have reference to two (or) more Independent variable. Eg:- $d^2y/dx^2 = 1/c^2 \, d^2y/dt^2$, $\frac{d^2y}{dx^2} + \frac{d^2y}{dt^2} = 0$.

Order of D.E :- Highest derivative in the given eq^n is called order of D.E. (2)

degree of D.E :- Highest power of highest derivative which occurs in D.E and it is free from radicals and fractions is called "degree of D.E".

Eg :- $\left(d^2y/dx^2\right)^3 + 5x\left(dy/dx\right)^5 + 6y = 0$ $\therefore$ order = 2 ; degree = 3.

Eg :- $\left(dy/dx\right)^2 = \cot x$ $\therefore$ order = 1 ; degree = 2.

Ex :- $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

$\Rightarrow y - x\frac{dy}{dx} = \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$

Sq. on b/s

$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$

$\Rightarrow y^2 - 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$

$\Rightarrow \left(\frac{dy}{dx}\right)^2(x^2-1) - 2xy\left(\frac{dy}{dx}\right) + (y^2-1) = 0$

order = 1 ; degree = 2

Formation of a differential Eq^{n}s :-

(1). Find the differential Eq^n for $y = ae^x + be^{-x}$.

Sol^n :- Given Eq^n is $y = ae^x + be^{-x}$ ——— (1)

Here a, b are arbitrary constants

diff^n (1) w.r.t 'x' :

$\frac{dy}{dx} = ae^x - be^{-x}$ ——— (2)

Again diff. w.r.to 'x'.

$$\frac{d^2y}{dx^2} = ae^x + be^{-x} \quad \text{——— (3)}$$

$$\frac{d^2y}{dx^2} = y \quad [\because (1)]$$

$$\Rightarrow \boxed{y'' - y = 0}$$

(2). Construct the D.E for $xy = ae^x + be^{-x} + x^2$

Sol:- Given $xy = ae^x + be^{-x} + x^2$

$$\Rightarrow (xy - x^2) = ae^x + be^{-x} \quad \text{——— (1).}$$

Here a, b are arbitrary constants

$$x\frac{dy}{dx} + y - 2x = ae^x - be^{-x} \quad \text{——— (2)}$$

Again diff. w.r.to 'x'

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} - 2 = ae^x + be^{-x}$$

$$xy'' + 2y' - 2 = xy - x^2 \quad [\because (1)]$$

$$\Rightarrow xy'' + 2y' - xy + x^2 - 2 = 0$$

//

Formation of differential Eqⁿs :-

To find the diff. eqⁿ corresponding to the function

$$f(x, y, c_1, c_2, \ldots c_n) = 0 \quad \text{——— (1).}$$

Eliminate the arbitrary constant $c_1, c_2, \ldots, c_n$ by diff. Eqⁿ(1) 'n' times. Here 'x' is an independent variable.

'y' is a dependent variable.

(4)

Diff. Eq^n(1) w.r.t 'x' 1st time, we get $f\left(x, y, \frac{dy}{dx}, c_1, c_2, \dots, c_n\right) = 0$ — (2)

Again diff. eq^n(2) 2nd time w.r.t 'x', we get

$$f\left(x, y, y', \frac{d^2y}{dx^2}, c_1, c_2, \dots, c_n\right) = 0 \quad \text{— (3)}$$

and so on n^{th} times diff. on Eq^n w.r.t 'x' from

$$f(x, y, y', y'', \dots, y^n, c_1, c_2, \dots, c_n) = 0 \quad \text{— } (n+1)$$

By solving Eq^n eliminate the arbitrary coefficient $c_1, c_2, \dots, c_n$.

then we get the diff. eq^n $f\left(y^n, y^{(n-1)}, \dots, y', y, x\right) = 0$

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \dots, \frac{dy}{dx}, y, x\right) = 0$$

(3). Find the D.E for $\log\left(\frac{y}{x}\right) = cx$

Sol^n:- Given Eq^n $\log(y/x) = cx \rightarrow (1)$.

$\Rightarrow \log y - \log x = cx$, where '$c$' is an arbitrary constant

Diff. w.r.to 'x'

$$\frac{1}{y}\frac{dy}{dx} - \frac{1}{x} = c$$

$$\frac{y'}{y} - \frac{1}{x} = \frac{\log\left(\frac{y}{x}\right)}{x} \quad [\because (1)]$$

$$\Rightarrow x\left(\frac{y'}{y} - \frac{1}{x}\right) = \log(y/x).$$

(4). Find the D.E for $\sin^{-1}x + \sin^{-1}y = c$.

Sol^n:-

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \quad //$$

5. Find the D.E for $y = Ae^{-2x} + Be^{5x}$ — (1).

Here A, B are arbitrary constants.

Diff. eqn (1) w.r.to 'x'.

$$\frac{dy}{dx} = Ae^{-2x}(-2) + Be^{5x}(5)$$

$$y' = -2Ae^{-2x} + 5Be^{5x} \quad \text{— (2).}$$

$$y' = -2Ae^{-2x} - 2Be^{5x} + 7Be^{5x}$$

$$= -2\left(Ae^{-2x} + Be^{5x}\right) + 7Be^{5x}$$

$$y' = -2y + 7Be^{5x} \quad \text{— (3)} \quad [\because (1)]$$

Again Diff. w.r.to 'x' eqn (3)

$$y'' = -2y' + 7Be^{5x}(5)$$

$$y'' + 2y' = 5\left(7Be^{5x}\right)$$

$$y'' + 2y' = 5\left(y' + 2y\right)$$

$$\Rightarrow y'' - 3y' - 10y = 0.$$

(or)

Given $y = Ae^{-2x} + Be^{5x}$ — (1)

Diff. (1) w.r.t 'x'

$$y' = -2Ae^{-2x} + 5Be^{5x} \quad \text{— (2).}$$

Again Diff. w.r.t 'x'.

$$y'' = 4Ae^{-2x} + 25Be^{5x} \quad \text{— (3).}$$

$$\begin{vmatrix} e^{-2x} & e^{5x} & -y \\ -2e^{-2x} & 5e^{5x} & -y' \\ 4e^{-2x} & 25e^{5x} & -y'' \end{vmatrix} = 0 \Rightarrow e^{-2x}\cdot e^{5x}\begin{vmatrix} 1 & 1 & -y \\ -2 & 5 & -y' \\ 4 & 25 & -y'' \end{vmatrix} = 0$$

$$\Rightarrow y'' - 3y' - 10y = 0$$

6. Find the D.E for $y^2 = (x-c)^3$ —— (1)

$\Rightarrow x - c = y^{2/3}$ —— (2)

Diff. Eq^n (1) w.r.t 'x'

$\Rightarrow 2y\frac{dy}{dx} = 3(x-c)^2$ —— (3)

[$\Rightarrow 2y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(2\frac{dy}{dx}\right) = 6(x-c)$

$\Rightarrow 2yy'' + (y')^2(2) = 6$]

$\Rightarrow 2yy' = 3y^{4/3}$ [$\because$ (2)]

$\Rightarrow 2y' = 3y^{4/3}y^{-1}$

$\Rightarrow 2y' = 3y^{1/3}$

$\Rightarrow 2y' - 3y^{1/3} = 0$.

7. Find the D.E for circle passing through origin and having centre on x-axis.

Sol: We know that Eq^n of circle with centre $(-g, -f)$ is

$x^2 + y^2 + 2gx + 2fy + c = 0$.

Since the centre is on x-axis and the circle passing through origin is $f = 0$ and $c = 0$.

Hence the given Eq^n family of circles is

$x^2 + y^2 + 2gx = 0$. —— (1)

where 'g' is a parameter

$2x + 2y\frac{dy}{dx} + 2g = 0 \Rightarrow g = -\left(x + y\frac{dy}{dx}\right)$ —— (2)

$$\Rightarrow x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)=0$$

$$\Rightarrow y^2-x^2-2xy\frac{dy}{dx}=0.$$

(8). Find the differential Eq^n of the family of parabolas having vertex at the origin and foci on Y-axis.

Sol:- $x^2=4ay$ where 'a' is a parameter. — (1)

Diff. w.r.t 'x':

$$2x=4a\frac{dy}{dx}$$

$$a=\frac{x}{2\left(\frac{dy}{dx}\right)} \quad \text{— (2).}$$

(2) in (1)

$$x^2=\frac{4x}{2\left(\frac{dy}{dx}\right)}y.$$

$$x\left(\frac{dy}{dx}\right)=2y$$

this is the required D.E of the given family of parabolas.

Differential Eq^ns of the 1st order and of the 1st degree :- An eq^n of the form $\frac{dy}{dx}=f(x,y)$ is called a D.E of 1st order and of 1st degree.

In general first order D.E can be classified as below:

(1). variables separable.

(2). Homogenous Eq^ns and Eq^ns reducible to homogenous form

(3). Exact Eq^ns and those which can be made exact by use of I.F.

(4). Linear Equations & Bernoulli's Equations. (8)

Variable separable method :— the D.E of the form $\frac{dy}{dx} = f(x,y)$. i.e $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (or) $g(y)\,dy = f(x)\,dx$ (or) (1) $f(x)\,dx - g(y)\,dy = 0$. where 'f' and 'g' are continuous functions of a single variable, then it is said to be of the form "variable-separable." integrating the (1), the general soln is

$$\int f(x)\,dx - \int g(y)\,dy = c,$$ where 'c' is any arbitrary constant.

(1) Solve the D.E $\sin^{-1}x\,dy + \frac{y\,dx}{\sqrt{1-x^2}} = 0$.

Sol: Given $\sin^{-1}x\,dy + \frac{y\,dx}{\sqrt{1-x^2}} = 0$.

$$\Rightarrow \sin^{-1}x\,dy = \frac{-y\,dx}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{y} = \frac{-dx}{\sin^{-1}x\sqrt{1-x^2}}$$

$$\Rightarrow \int \frac{1}{y}\,dy = -\int \frac{\left(\frac{dx}{\sqrt{1-x^2}}\right)}{\sin^{-1}x}$$

$$\Rightarrow \log y = -\int \left(\frac{1}{\sqrt{1-x^2}}\right) \Big/ \sin^{-1}x \; dx.$$

$$\Rightarrow \log y = -\log|\sin^{-1}x| + \log c$$

$$\Rightarrow \log y = \log\left(\frac{c}{\sin^{-1}x}\right)$$

$$\Rightarrow y = \frac{c}{\sin^{-1}x} \Rightarrow \boxed{y\sin^{-1}x = c}.$$

(2) Solve the D.E $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Sol:- $\Rightarrow \frac{dy}{dx} = e^{x-y} + x^2 e^{-y} \Rightarrow \frac{dy}{dx} = e^{x} e^{-y} + x^2 e^{-y}$

$\Rightarrow \frac{dy}{dx} = e^{-y}(e^{x}+x^2) \rightarrow \int e^{y} dy = \int (e^{x}+x^2) dx$

$\Rightarrow e^{y} = e^{x} + \frac{x^3}{3} + c$

$\Rightarrow e^{y} - e^{x} - \frac{x^3}{3} = c$ //

Homogenous Eqn (or) function :- A function $f(x,y)$ is said to be "Homogenous function" in 'x' & 'y' of degree 'n', if $f(kx, ky) = k^n f(x,y)\ \forall k$, where '$n$' is a constant.

Ex:- (i). $f(x,y) = \frac{x^2+y^2}{x^3+y^3}$

$f(kx, ky) = \frac{k^2x^2+k^2y^2}{k^3x^3+k^3y^3}$

$= k^{-1} f(x,y)$

(ii). $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$

$f(kx, ky) = \frac{k^3x^3+k^3y^3}{k^2x^2+k^2y^2} = k f(x,y)$.

(iii). $f(x,y) = \cos x + \tan y$

$f(kx, ky) = \cos(kx) + \tan(ky) \neq k^n f(x,y)$.

$\therefore f(x,y)$ is not a homogenous function.

Homogenous Diffn Eqn :- A D.E $\frac{dy}{dx} = f(x,y)$ is of 1st order and 1st degree is called a Homogenous Diffn Eqn, if the function of x, y is a Homogenous function of degree '0' in 'x' & 'y'. Eg:- $f(x,y) = 2xy/x^2+y^2$

$f(kx, ky) = 2(kx)(ky)/k^2x^2+k^2y^2 = k^0 f(x,y)$.

Non Homogeneous D.E (N.H.D.E) :- N.H.D.E of 1st order and 1st degree in 'x' and 'y' if a_1, b_1, c_1 and a_2, b_2, c_2 are constants and atleast one of 'c_1' and 'c_2' is not zero, then $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is called a N.H.D.E.

Procedure for solving N.H.D.E :-

Consider $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is an eqn ——— (1)

case (1) :- If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Put $\left.\begin{array}{l} x = X+h \Rightarrow X = x-h \\ y = Y+k \Rightarrow Y = y-k \end{array}\right\}$ ——— (2).

then $\frac{dy}{dx} = \frac{dY}{dX}$ ——— (3).

sub,, (2) & (3) in (1), we get—

$$\frac{dY}{dX} = \frac{a_1(X+h)+b_1(Y+k)+c_1}{a_2(X+h)+b_2(Y+k)+c_2}$$

$$\frac{dY}{dX} = \frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$$

If $a_1h+b_1k+c_1 = 0$ &

$a_2h+b_2k+c_2 = 0$ then by solving these two Eqns we get the values of 'h' and 'k'. then the reduced D.E is

$\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}$, which is in the form of H.D.E

Procedure for solving D.E :-

(1). Consider the given D.E as Eqⁿ(1).

(2). check whether it is a Homogenous D.E (or) not.

(3) Assume $\boxed{y = vx}$ as Eqⁿ(2)

(4). Differentiate Eqⁿ(2) w.r.to 'x' on b.s, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{---} (3).$$

(5). Sub., Eqⁿ(2) & (3) in (1) and using variable separable method, we can get the solⁿ of the D.E.

Problems :-

(1). Solve $x\frac{dy}{dx} = y + x \cdot e^{y/x}$

Solⁿ:- Let $f(x,y) = \frac{dy}{dx} = \frac{y}{x} + e^{y/x}$ — (1)

$$f(kx, ky) = \frac{ky}{kx} + e^{ky/kx}$$

$$= \frac{y}{x} + e^{y/x}$$

$\therefore f(kx, ky) = k^0 f(x,y)$

$\therefore f(x,y)$ is a homogenous D.E.

Let us Assume $y = vx$ — (2) $\Rightarrow (v = y/x)$

Diff. (2) on b.s w.r.to 'x'.

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{---} (3)$$

Sub. (2) & (3) in (1).

$\Rightarrow v + x\frac{dv}{dx} = v + e^{v}$

$\Rightarrow x\frac{dv}{dx} = e^{v}$

$\Rightarrow \frac{dv}{e^{v}} = \frac{1}{x}dx$

$\Rightarrow \int \frac{1}{e^{v}}dv = \int 1/x\, dx$

$\Rightarrow \int e^{-v}dv = \log x + \log c$

$\Rightarrow \frac{e^{-v}}{(-1)} = \log(cx)$

$\Rightarrow -e^{-v} = \log(cx)$

$\Rightarrow -e^{-y/x} = \log(xc)$

$\boxed{\because y = vx \\ \Rightarrow v = y/x}$

$\Rightarrow \log(xc) + e^{-y/x} = 0.$

and can be solve by using the procedure of H.D.E (11)
at the end, put $X = x - h$ and $Y = y - k$, which is the required soln for given N.H.D.E.

<u>case (2)</u> :- Suppose $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Take $\frac{a_2}{a_1} = \frac{b_2}{b_1} = k$ where 'k' is a constant.

$\Rightarrow a_2 = a_1 k \; ; \; b_2 = b_1 k$

Sub.. a_2, b_2 in (1), then we get

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{k(a_1 x + b_1 y) + c_2}.$$

this can be reduced to variable separable form by writing $\boxed{z = a_1 x + b_1 y}$.

<u>case (3)</u>:- Suppose $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

then take $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = \frac{1}{m}$

Sub.. in Eqn (1), we get

$$\frac{dy}{dx} = m$$

By separating the variables

$$dy = m\,dx.$$

Integrating on both sides

$\int dy = m \int dx \Rightarrow \boxed{y = mx + c}$. which is a st. line.

1.29

①. $(x+y-1)\dfrac{dy}{dx} = x-y+2$

Sol:- $\dfrac{dy}{dx} = \dfrac{x-y+2}{x+y-1}$ — (1) Here $a_1=1$; $b_1=-1$, $c_1=2$
$a_2=1$; $b_2=1$; $c_2=-1$

$\therefore \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

Put $x = X+h$; $y = Y+k$.

then $\dfrac{dy}{dx} = \dfrac{dY}{dX}$.

$$(1) \Rightarrow \frac{dY}{dX} = \frac{(X+h)-(Y+k)+2}{(X+h)+(Y+k)-1} = \frac{X-Y+(h-k+2)}{X+Y+(h+k-1)} \quad \text{— (2)}$$

Take $h-k+2=0$
$h+k-1=0$

Solving these two Eq's, we get $\boxed{h=-1/2 \quad k=3/2}$

$$(2) \Rightarrow \frac{dY}{dX} = \frac{X-Y}{X+Y} \quad \text{— (3)}$$

Here $f(X,Y) = X-Y/X+Y$

Now check here $f(kX,kY) = \dfrac{kX-kY}{kX+kY} = \left(\dfrac{X-Y}{X+Y}\right)k^0$.

$\therefore f(kX,kY) = k^n f(X,Y)$

where $n=0$.

$\therefore f(X,Y)$ is a homogeneous function.

Put $Y = vX$ — (4) $\Rightarrow \boxed{v = Y/X}$

$$\frac{dY}{dX} = v + X\frac{dv}{dX} \quad \text{— (5)}.$$

$$V + X\frac{dV}{dX} = \frac{X - VX}{X + VX} = \frac{1-V}{1+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1-V}{1+V} - V$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1-V-V-V^2}{1+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1-2V-V^2}{1+V} \Rightarrow X\frac{dV}{dX} = \frac{-(V^2+2V-1)}{1+V}$$

$$\Rightarrow \int \frac{1+V}{-1+2V+V^2}\,dV = -\int \frac{1}{X}\,dX$$

$$\Rightarrow \frac{1}{2}\int \frac{2+2V}{V^2+2V-1}\,dV = -\int \frac{1}{X}\,dX$$

$$\Rightarrow \frac{1}{2}\log|(V^2+2V-1)| = -\log|X| + \log|c|$$

$$\Rightarrow \log|V^2+2V-1| = -2\log X + 2\log c$$

$$\Rightarrow \quad = -\log X^2 + \log c^2$$

$$\Rightarrow \quad = \log\left(\frac{c^2}{X^2}\right)$$

$$\Rightarrow \frac{Y^2}{X^2} + 2\frac{Y}{X} - 1 = \frac{c^2}{X^2}$$

$$\Rightarrow Y^2 + 2XY - X^2 = c^2$$

But $X = x - h = x + 1/2$

$Y = y - k = y - 3/2$

$\Rightarrow$ G.S is $\left(y - 3/2\right)^2 + 2\left(x + 1/2\right)\left(y - 3/2\right) - \left(x + 1/2\right)^2 = C.$

(2). Solve $\dfrac{x+2y+1}{2x+4y+3} = \dfrac{dy}{dx}$.

(14).

Sol^n:- $\dfrac{dy}{dx} = \dfrac{x+2y+1}{2(x+2y)+3}$ ——— (1)

Here $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$

Let $x+2y = z$. ——— (2)

$1+2\dfrac{dy}{dx} = \dfrac{dz}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{\frac{dz}{dx}-1}{2}$ ——— (3)

(1) $\Rightarrow \dfrac{\frac{dz}{dx}-1}{2} = \dfrac{z+1}{2z+3} \Rightarrow \dfrac{dz}{dx} = \dfrac{2z+2}{2z+3}+1$

$\Rightarrow \dfrac{dz}{dx} = \dfrac{2z+2+2z+3}{2z+3}$

$\Rightarrow \dfrac{dz}{dx} = \dfrac{4z+5}{2z+3}$

$\Rightarrow \displaystyle\int \frac{(2z+3)}{4z+5}\,dz = \int dx$

$\Rightarrow \dfrac{1}{2}\displaystyle\int \frac{4z+6}{4z+5}\,dz = \int dx$

$\Rightarrow \dfrac{1}{2}\displaystyle\int \frac{(4z+5)+1}{4z+5}\,dz = x + c$

$\Rightarrow \dfrac{1}{2}z + \dfrac{1}{2}\dfrac{\log(4z+5)}{4} = x + c$

$\Rightarrow 4z + \log(4z+5) = 8x + 8c.$

G.S is

$\Rightarrow 4(x+2y) + \log(4x+8y+5) = 8x + 8c$ //

Exact D.E :- Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be a 1st order and 1st degree D.E, where M, N are real valued functions for some x, y. then the Eqn $Mdx+Ndy=0$ is said to be an Exact D.E (E.D.E) if ∃ a fn 'f' ∋

$\frac{\partial f}{\partial x} = M$; $\frac{\partial f}{\partial y} = N$.

Eg:- the D.E $2xy\,dx + x^2 dy = 0$ is an Exact D.E.

Since ∃ a fn $f(x,y) = x^2y$ ∋ $\frac{\partial f}{\partial x} = 2xy = M$ &

$\frac{\partial f}{\partial y} = x^2 = N$.

Eg:- $xdy+ydx=0$ is obtained by differentiating $xy=c$. ∴ This is an exact d.E.

Conditions for Exactness :- If $M(x,y)$ and $N(x,y)$ are two real valued functions, which have continuous partial derivative then necessary condition and sufficient condition for A D.E is of the form $Mdx+Ndy=0$ is called E.D.E

i.e, $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$.

Soln of an E.D.E :- $\int M dx + \int N dy = c$.
y constant   donot take x-terms

Problems :- ①. Solve the D.E $\frac{dy}{dx} = \frac{-(x+2y-1)}{2x+y-2}$.

Sol:- Given Eqn can be written as

$(2x+y-2)\,dy = -(x+2y-1)\,dx$

$\Rightarrow (x+2y-1)\,dx + (2x+y-2)\,dy = 0$

which is in the form of $M\,dx + N\,dy = 0$.

where $M = x+2y-1$; $N = 2x+y-2$

$\frac{\partial M}{\partial y} = 2$; $\frac{\partial N}{\partial x} = 2$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ G.S is $\int M\,dx + \int N\,dy = C$

$\Rightarrow \int (x+2y-1)\,dx + \int (2x+y-2)\,dy = C$

$\Rightarrow \frac{x^2}{2} + 2yx - x + \frac{y^2}{2} - 2y = C$

$\Rightarrow x^2 + 4yx - 2x - 4y + y^2 = 2C$ //.

(2). Solve the D.E $\left(xe^{xy} + 2y\right)\frac{dy}{dx} + ye^{xy} = 0$

Sol:- $\underline{\left(xe^{xy}+2y\right)\frac{dy}{dx}} + ye^{xy}\,dx = 0$

$\Rightarrow \left(xe^{xy}+2y\right)dy + ye^{xy}\,dx = 0$

$\Rightarrow \left(ye^{xy}\right)dx + \left(xe^{xy}+2y\right)dy = 0$.

which is in the form of $M\,dx + N\,dy = 0$

where $M = ye^{xy}$; $N = xe^{xy} + 2y$.

$\frac{\partial M}{\partial y} = e^{xy} + ye^{xy}(x)$; $\frac{\partial N}{\partial x} = xe^{xy}(y) + e^{xy} + 0$

$= e^{xy}(1+xy)$ $\qquad$ $= e^{xy}(xy+1)$

$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$\therefore$ G.S is $\int (M) dx + \int N dy = c$

$\int y e^{xy} dx + \int 2y\, dy = c$

$\Rightarrow \cancel{y} \frac{e^{xy}}{\cancel{y}} + \cancel{2} \frac{y^2}{\cancel{2}} = c$

$\Rightarrow \boxed{e^{xy} + y^2 = c}$.

(3). $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Sol:- $\frac{dy}{dx} = -\left(\frac{y\cos x + \sin y + y}{\sin x + x\cos y + x}\right)$

$\Rightarrow -(y\cos x + \sin y + y)dx = (\sin x + x\cos y + x)\, dy$

$\Rightarrow \underbrace{(y\cos x + \sin y + y)}_{(M)} dx + \underbrace{(\sin x + x\cos y + x)}_{N} dy = 0.$

where $M = y\cos x + \sin y + y$; $N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + 1 + \cos y$; $\frac{\partial N}{\partial x} = 1 + (\cos y) + \cos x$

$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

G.S is $\int (M) dx + \int N dy = c$

$\Rightarrow \int (y\cos x + \sin y + y)\, dx + \int (\sin x + x\cos y + x)\, dy = c$

$\Rightarrow y\sin x + \sin y(x) + y(x) + 0 = c$

$\Rightarrow y\sin x + x\sin y + xy = c$ //

4. Solve $x^3\sec^2 y\frac{dy}{dx}+3x^2\tan y=\cos x$.

Sol: $x^3\sec^2 y\,dy+(3x^2\tan y)\,dx=\cos x\,dx$

$\Rightarrow (3x^2\tan y-\cos x)\,dx+x^3\sec^2 y\,dy=0.$

$M\,dx+N\,dy=0.$

where $M=3x^2\tan y-\cos x$; $N=x^3\sec^2 y$

$\frac{\partial M}{\partial y}=3x^2\sec^2 y+0$; $\frac{\partial N}{\partial x}=\sec^2 y(3x^2)$

$=3x^2\sec^2 y$; $=3x^2\sec^2 y$

$\therefore \boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$

$\therefore$ G.S is $\int M\,dx+\int N\,dy=C$

$\Rightarrow \int(3x^2\tan y-\cos x)\,dx+\int(x^3\sec^2 y)\,dy=C$

$\Rightarrow 3\tan y\left(\frac{x^3}{3}\right)-\sin x+0=C$

$\Rightarrow x^3\tan y-\sin x=C.$

5. $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0$

Sol: where $M=(e^y+1)\cos x$; $N=e^y\sin x$

$\frac{\partial M}{\partial y}=\cos x(e^y)$; $\frac{\partial N}{\partial x}=e^y\cos x$

$\therefore \boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$

$\therefore$ G.S is $\int M\,dx+\int N\,dy=C$

$\Rightarrow \int(e^y+1)\cos x\,dx+\int e^y\sin x\,dy=C$

$\Rightarrow \int e^y\cos x\,dx+\int\cos x\,dx+0=C$

$\Rightarrow e^y\sin x+\sin x=C$ //.

6. $2xy\,dy-(x^2-y^2+1)\,dx=0$

A: $(-x^3/3+y^2x-x=c)$

7. $(x^2-y^2)\,dx=2xy\,dy$

A: $(x^3/3-y^2x=c)$

8.

Non Exact D.E :- The D.E $Mdx + Ndy = 0$ is called Non exact D.E if $\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$.

Integrating Factor :- Let $\underbrace{Mdx + Ndy = 0}_{(1)}$ be not an exact D.E, we can made exact by multiplying Eq^n (1) is a suitable function $u(x,y) \neq 0$, then $u(x,y)$ is called an Integrating factor (or) Integrating factor of $Mdx + Ndy = 0$.

Method (1) :- To find an Integrating factor of $Mdx+Ndy=0$ by using some formulae.

1. $d(xy) = xdy + ydx$
2. $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$
3. $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$
4. $d\left(\frac{x^2+y^2}{2}\right) = xdx + ydy$
5. $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$
6. $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{ydx - xdy}{xy}$
7. $d\left[\tan^{-1}(y/x)\right] = \frac{xdy - ydx}{x^2+y^2}$

8. $d\left[\tan^{-1}(x/y)\right] = \dfrac{y\,dx - x\,dy}{x^2+y^2}$

9. $d\left[\log(xy)\right] = \dfrac{y\,dx + x\,dy}{xy}$

10. $d\left[\log(x^2+y^2)\right] = \dfrac{2(x\,dx + y\,dy)}{x^2+y^2}$

11. $d\left(\dfrac{e^x}{y}\right) = \dfrac{y e^x dx - e^x dy}{y^2}$

1. Solve $x\,dx + y\,dy = \dfrac{x\,dy - y\,dx}{x^2+y^2}$

Sol:- $d\left(\dfrac{x^2+y^2}{2}\right) = d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]$

Integrating on b.s

$\int d\left(\dfrac{x^2+y^2}{2}\right) = \int d\left[\tan^{-1}(y/x)\right]$

$\Rightarrow \dfrac{x^2+y^2}{2} = \tan^{-1}(y/x) + c$ //

2. $x\,dx + y\,dy = a^2\left(\dfrac{x\,dy - y\,dx}{x^2+y^2}\right)$

Sol: W.K.T from D.E formulae is

$d\left(\dfrac{x^2+y^2}{2}\right) = a^2\, d\left[\tan^{-1}(y/x)\right]$

$\int d\left(\dfrac{x^2+y^2}{2}\right) = a^2 \int d\left[\tan^{-1}(y/x)\right]$

$\Rightarrow \dfrac{x^2+y^2}{2} = a^2 \tan^{-1}(y/x) + c$ //

③. $(y-x^2)\,dx + (x^2\cot y - x)\,dy = 0$

Sol:- $y\,dx - x^2\,dx + x^2\cot y\,dy - x\,dy = 0$

$\Rightarrow (y\,dx - x\,dy) = x^2\,dx - x^2\cot y\,dy$

$\Rightarrow \dfrac{y\,dx - x\,dy}{x^2} = dx - \cot y\,dy$ ($\because$ Divide by x^2 on b.s)

$\Rightarrow -\left(\dfrac{x\,dy - y\,dx}{x^2}\right) = dx - \cot y\,dy.$

$\Rightarrow -\int d\left(\dfrac{y}{x}\right) = \int dx - \int \cot y\,dy$

$\Rightarrow -\dfrac{y}{x} = x - \log|\sin y| + c$

$\Rightarrow x + \dfrac{y}{x} - \log\sin y = c.$

④. $x\,dy - y\,dx = xy^2\,dx$

Sol:- $x\,dy - y\,dx = xy^2\,dx$

divide by 'y^2' on b.s

$\Rightarrow \dfrac{-(-x\,dy + y\,dx)}{y^2} = \dfrac{xy^2}{y^2}\,dx$

$\Rightarrow -d\left(\dfrac{x}{y}\right) = x\,dx$

$\Rightarrow -\left(\dfrac{x}{y}\right) = \dfrac{x^2}{2} + C$

$\Rightarrow \dfrac{x^2}{2} + \dfrac{x}{y} = C$ //

(22)

(5) $y(2x^2y+e^x)\,dx = (e^x+y^3)\,dy$.

Sol:- $2x^2y^2dx + e^xy\,dx = e^xdy + y^3dy$

$$\Rightarrow 2x^2dx + \frac{ye^xdx - e^xdy}{y^2} - y\,dy = 0 \quad (\because \text{Dividing with } y^2)$$

$$\Rightarrow 2x^2dx + d\left(\frac{e^x}{y}\right) - y\,dy = 0.$$

$$\Rightarrow \frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c.$$

which is the required G.S.

(6). $$\frac{y(xy+e^x)dx - e^xdy}{y^2} = 0$$

Sol:- $$\frac{(xy^2+e^xy)\cdot dx - e^xdy}{y^2} = 0$$

$$\Rightarrow \frac{xy^2}{y^2}dx + \left(\frac{e^xy\,dx - e^xdy}{y^2}\right) = 0$$

$$\Rightarrow x\,dx + d\left(\frac{e^x}{y}\right) = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{e^x}{y} = c$$ //

(7). $y\,dx + x\,dy + xy(y\,dx - x\,dy) = 0$

Sol:- $d(xy) + xy^2dx - x^2y\,dy = 0$

Dividing with x^2y^2

$$\frac{d(xy)}{x^2y^2} + \frac{1}{x}dx - \frac{1}{y}dy = 0.$$

$$-\frac{1}{xy} + \log\left(\frac{x}{y}\right) = \log c$$ //

(23)

Method (2) :- To find the I.F of $Mdx+Ndy=0$.

If $M(x,y)\,dx+N(x,y)\,dy=0$ is a homogenous D.E & $Mx+Ny\neq 0$ and $Mdx+Ndy=0$ is not an E.D.E, then $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx+Ndy=0$.

(1). Solve $x^2y\,dx-(x^3+y^3)\,dy=0$ — (1)

Sol:- $M=x^2y$; $N=-(x^3+y^3)$

$\frac{\partial M}{\partial y}=x^2$; $\frac{\partial N}{\partial x}=-3x^2$

$\therefore \boxed{\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}}$

$\therefore$ Given D.E is not an E.D.E.

$$f(x,y)=\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}.$$

$$f(kx,ky)=\frac{k^2x^2ky}{k^3x^3+k^3y^3}=k^0\left(\frac{x^2y}{x^3+y^3}\right)=k^0f(x,y).$$

$\therefore$ It is a homogenous function and it is H.D.E.

& $Mx+Ny=x^3y-x^3y-y^4=-y^4\neq 0$.

$$\therefore \text{I.F}=\frac{1}{Mx+Ny}=\frac{-1}{y^4}.$$

multiplying Eqn(1) with $\frac{-1}{y^4}$, we get—

$$\frac{+x^2}{-y^3}dx+\frac{(x^3+y^3)}{y^4}dy=0 \quad —(2).$$

$(M_1dx+N_1dy=0)$

where $M_1 = \frac{-x^2}{y^3}$, $N_1 = \frac{x^3}{y^4} + \frac{1}{y}$

$\frac{\partial M_1}{\partial y} = -x^2(-3)y^{-4}$; $\frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4}$

$= \frac{3x^2}{y^4}$

$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$

$\therefore$ Eq^n(2) is an E.D.E.

$\Rightarrow \int M_1(x,y)\,dx + \int N_1(x,y)\,dy = c$

$\Rightarrow \int \frac{-x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$

$\Rightarrow \frac{-x^3}{3y^3} + \log y = c.$

(2). Solve $xy\,dx - (x^2+2y^2)\,dy = 0.$

Sol:- where $M = xy$; $N = -(x^2+2y^2)$.

$\frac{\partial M}{\partial y} = x$; $\frac{\partial N}{\partial x} = -2x.$

$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$

$\therefore$ Given D.E is not an E.D.E.

$f(x,y) = \frac{dy}{dx} = \frac{xy}{x^2+2y^2}$

$f(kx, ky) = k^0 f(x,y)$

It is a H.D.E.

$Mx+Ny = -2y^3 \neq 0$. & $I.F = \frac{1}{Mx+Ny} = \frac{1}{-2y^3}$

Multiplying Eqn(1) with $\frac{-1}{2y^3}$, we get –

$$\Rightarrow \frac{-xy}{2y^3}dx + \left(\frac{x^2+2y^2}{2y^3}\right)dy = 0.$$

$$\Rightarrow \underset{M_1}{\frac{-x}{2y^2}}dx + \underset{N_1}{\left(\frac{x^2}{2y^3}+\frac{1}{y}\right)}dy = 0$$

& $M_1 = \frac{-x}{2y^2}$; $N_1 = \frac{x^2}{2y^3} + \frac{1}{y}$

$$\frac{\partial M_1}{\partial y} = -\frac{x}{2}(-2)y^{-3} \quad ; \quad \frac{\partial N_1}{\partial x} = \frac{2x}{2y^3}$$

$$= \frac{x}{y^3} \qquad\qquad = x/y^3.$$

$$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

$\therefore$ G.S is
$$\int M_1 dx + \int N_1 dy = C$$

$$\Rightarrow \int\left(\frac{-x}{2y^2}\right)dx + \int \frac{1}{y}dy = C$$

$$\Rightarrow \frac{-1}{2y^2}\left(\frac{x^2}{2}\right) + \log y = C$$

$$\Rightarrow \frac{-x^2}{4y^2} + \log y = C$$

H.W
(3). Solve $(x^2y-2xy^2)dx - (x^3-3x^2y)dy = 0$ [A: $\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = C$]

H.W
(4). $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$ [A:- $\log(x^2+y^2)^{1/2} + \tan^{-1}(y/x) = C$]

Method (3) :- To find an Integrating Factor of

$$Mdx + Ndy = 0.$$

If the Eqn $Mdx + Ndy = 0$ is of the form not N.H.D.E and it is in the form of $y\,f(x,y)\,dx + x\,g(x,y)\,dy = 0$ then the I.F is $\frac{1}{Mx - Ny}$ where $Mx - Ny \neq 0$.

(1).

Solve $y\left(x^2y^2 + 2\right)dx + x\left(2 - 2x^2y^2\right)dy = 0$ ——(1)

Sol:- This is of the form $Mdx + Ndy = 0$.

where $M = y\left(x^2y^2 + 2\right)$; $N = x\left(2 - 2x^2y^2\right)$

we have $\frac{\partial M}{\partial y} = 3x^2y^2 + 2$; $\frac{\partial N}{\partial x} = 2 - 6x^2y^2$

$$\therefore \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

$\therefore$ (1) is not a E.D.E.

& $f(x,y) = \frac{dy}{dx} = \frac{-y\left(x^2y^2 + 2\right)}{x\left(2 - 2x^2y^2\right)}$

$$f(kx, ky) = \frac{-\cancel{k}y\left(k^2x^2k^2y^2 + 2\right)}{\cancel{k}x\left(2 - 2k^2x^2k^2y^2\right)} = \frac{-y}{x}\left(\frac{k^4x^2y^2 + 2}{2 - 2k^4x^2y^2}\right)$$

$$f(kx, ky) \neq k^0 f(x,y)$$

$\therefore$ (1) is not a H.D.E.

(1), It is in the form of

$$y\,f(x,y)\,dx + x\,g(x,y)\,dy = 0.$$

and $Mx - Ny = 3x^3y^3 \neq 0.$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}.$$

Multiplying Eqⁿ(1) with I.F

$$\frac{y(x^2y^2+2)}{3x^3y^3}\,dx + \frac{x(2-2x^2y^2)\,dy}{3x^3y^3} = 0.$$

$$\rightarrow \underbrace{\frac{x^2y^2+2}{3x^3y^2}}_{M_1}\,dx + \underbrace{\frac{2-2x^2y^2}{3x^2y^3}}_{N_1}\,dy = 0 \quad \text{—— (2).}$$

where $M_1 = \dfrac{x^2y^2+2}{3x^3y^2}$; $N_1 = \dfrac{2-2x^2y^2}{3x^2y^3}$

$$\frac{\partial M_1}{\partial y} = \frac{1}{3x^3}\left[\frac{y^2(2x^2y) - (x^2y^2+2)\,2y}{y^4}\right]$$

$$= \frac{1}{3x^3y^4}\left(\cancel{2x^2y^3} - \cancel{2y^3x^2} - 4y\right)$$

$$= \frac{-4y}{3x^3y^4} = \frac{-4}{3x^3y^3}.$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{3y^3}\left[\frac{x^2(-4xy^2) - (2-2x^2y^2)(2x)}{x^4}\right]$$

$$= \frac{1}{3y^3x^4}\left[-\cancel{4xy^2x^2} - 4x + \cancel{4x^3y^2}\right.$$

$$= -\frac{4}{3y^3x^3}$$

$$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}} \qquad \therefore \text{(2) is an E.D.E.}$$

G.S is

$\Rightarrow \int M_1 dx + \int N_1 dy = c$

$\Rightarrow \int\left(\dfrac{x^2y^2+2}{3x^3y^2}\right)dx + \int\left(\dfrac{2-2x^2y^2}{3x^2y^3}\right)dy = c$

$\Rightarrow \dfrac{1}{3y^2}\int\left(\dfrac{x^2y^2+2}{x^3}\right)dx + \int\dfrac{2}{3x^2y^3}dy - \int\dfrac{2x^2y^2}{3x^2y^3}dy = c$

$\Rightarrow \dfrac{y^2}{3y^2}\int\dfrac{1}{x}dx + \dfrac{2}{3y^2}\int\dfrac{1}{x^3}dx + 0 - \dfrac{2}{3}\int\dfrac{1}{y}dy = c$

$\Rightarrow \dfrac{1}{3}\log|x| + \dfrac{2}{3y^2}\left(\dfrac{x^{-3+1}}{-3+1}\right) - \dfrac{2}{3}\log y = c$

$\Rightarrow \dfrac{1}{3}\log|x| - \dfrac{1}{3y^2x^2} - \dfrac{2}{3}\log y = c.$

which is the general soln of (1) //

(2). solve $y(1+xy)dx + x(1-xy)dy = 0.$ —— (1)

$M = y + xy^2$; $N = x - x^2y$.

$\dfrac{\partial M}{\partial y} = 1+2xy$; $\dfrac{\partial N}{\partial x} = 1-2xy$ $\quad \therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}.$

$\Rightarrow f(x,y) = \dfrac{dy}{dx} = -\dfrac{y(1+xy)}{x(1-xy)} = \dfrac{-y-xy^2}{x-x^2y}$

$f(kx,ky) = \dfrac{-ky-kxk^2y^2}{kx-k^2x^2ky} = \dfrac{-ky-k^3xy^2}{kx-k^3x^2y} \neq k^0 f(x,y).$

$\therefore$ (1) is Not a Homogeneous D.E & not a E.D.

(1) is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0.$

$Mx - Ny = y\cancel{x} + x^2y^2 - \cancel{xy} + x^2y^2 = 2x^2y^2 \neq 0.$

$I.F = \frac{1}{Mx+Ny} = \frac{1}{2x^2y^2}$

multiplying (1) with I.F is, $\frac{y(1+xy)}{2x^2y^2}dx + \frac{x(1-xy)}{2x^2y^2}dy = 0$

$\Rightarrow \underbrace{\left(\frac{1+xy}{2x^2y}\right)}_{M_1}dx + \underbrace{\left(\frac{1-xy}{2y^2x}\right)}_{N_1}dy = 0. \quad \text{---} (2)$

where $M_1 = \frac{1+xy}{2x^2y}$; $N_1 = \frac{1-xy}{2y^2x}$

$\frac{\partial M_1}{\partial y} = \frac{1}{2x^2}\left[\frac{(y)(x) - (1+xy)}{y^2}\right]$; $\frac{\partial N_1}{\partial x} = \frac{1}{2y^2}\left[\frac{(x)(-y) - (1-xy)}{x^2}\right]$

$= \frac{1}{2x^2y^2}[yx - 1 - xy]$; $\frac{\partial N_1}{\partial x} = \frac{-\cancel{xy} - 1 + \cancel{xy}}{2y^2x^2}$

$= \frac{-1}{2x^2y^2}$. $\quad = \frac{-1}{2y^2x^2}$

$\therefore$ G.S is $\int M_1 dx + \int N_1 dy = C$ $\quad \boxed{\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$

$\Rightarrow \int\left(\frac{1+xy}{2x^2y}\right)dx + \int\frac{1-xy}{2y^2x}dy = C.$

$\Rightarrow \frac{1}{2y}\int\left(\frac{1+xy}{x^2}\right)dx + \int\frac{1}{2y^2x} - \int\frac{\cancel{x}y}{2y^2\cancel{x}}dy = C$

$\Rightarrow \frac{1}{2y}\int\frac{1}{x^2}dx + \frac{1}{2y}\int\frac{xy}{x^2}dx + 0 - \frac{1}{2}\int\frac{1}{y}dy = C$

$\Rightarrow \frac{1}{2y}\left(\frac{x^{-2+1}}{-2+1}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = C.$

$\Rightarrow -\frac{1}{yx} + \log x - \log y = C \Rightarrow \log(x/y) - \frac{1}{yx} = C$

Method (4) :- To find an integrating factor of $Mdx+Ndy=0$:-

If the Given D.E $M(x,y)\,dx+N(x,y)\,dy=0$ is Non Exact and non-Homogenous and not in the form of $y\,f(x,y)\,dx+x\,g(x,y)\,dy=0$ then the I.F is "$e^{\int f(x)dx}$" where

$$f(x)=\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]$$ and $f(x)$ is a continuous single variable function.

Note :- In this method, the function $N(x,y)$ is compared with the function $M(x,y)$.

Q. Solve $(x^3-2y^2)\,dx+2xy\,dy=0$

Sol:- The given D.E is $(x^3-2y^2)dx+(2xy)dy=0$ —— (1)

It is in the form of $Mdx+Ndy=0$

where $M=x^3-2y^2$; $N=2xy$.

$$\frac{\partial M}{\partial y}=-4y \quad ; \quad \frac{\partial N}{\partial x}=2y$$

$$\therefore \boxed{\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}}$$

Let $f(x,y)=\dfrac{dy}{dx}=\dfrac{-(x^3-2y^2)}{2xy}$.

$$f(kx,ky)=\frac{-(k^3x^3-2k^2y^2)}{2kx\cdot ky}\neq k^0 f(x,y).$$

$\therefore$ (1) is not Exact and non-Homogeneous D.E and not in the form of $y\,f(x,y)\,dx+g(x,y)\cdot x=0$ then check can find -

$$= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$= \frac{1}{2xy}\left(-4y - 2y\right) = -3/x = f(x).$$

$$\therefore I.F = e^{\int f(x)dx} = e^{-3/x\,dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3} = 1/x^3.$$

multiply (1) with $I.F = 1/x^3$.

$$\Rightarrow \left(\frac{x^3 - 2y^2}{x^3}\right)dx + \frac{2xy}{x^3}dy = 0$$

$$\Rightarrow \left(1 - \frac{2y^2}{x^3}\right)dx + \left(\frac{2y}{x^2}\right)dy = 0 \quad \text{——— (2)}$$

It is in the form of $M_1 dx + N_1 dy = 0$.

where $M_1 = 1 - \frac{2y^2}{x^3}$; $N_1 = \frac{2y}{x^2}$

$$\frac{\partial M_1}{\partial y} = -\frac{4y}{x^3} \; ; \; \frac{\partial N_1}{\partial x} = 2y(-2)x^{-3} = -\frac{4y}{x^3}.$$

$$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

$\therefore$ Eq^n(2) is an Exact D.E.

G.S is $\underset{y\text{-constant}}{\int M_1 dx} + \underset{\text{do not take 'x' terms}}{\int N_1 dy} = c$

$$\Rightarrow \int 1\,dx - 2y^2\int \frac{1}{x^3}dx + 0 = c \Rightarrow x - 2y^2\frac{x^{-2}}{-2} = c$$

$$\Rightarrow x + x^{-2}y^2 = c$$

$$\Rightarrow x + \frac{y^2}{x^2} = c$$

$$\Rightarrow \boxed{x^3 + y^2 = cx^2}$$

② Solve. $(x^2+y^2+2x)dx + 2y\,dy = 0$. (A:- $e^x(x^2+y^2) = c$)

③. Solve. $(x^3 - 2y^2)dx + 2xy\,dy = 0$. (A:- $x^3 + y^2 = cx^2$)

Method (5) :- To find an integrating factor of

$Mdx + Ndy = 0$.

If the given D.E $M(x,y)dx + N(x,y)dy = 0$ is not Exact not a H.D.E and not of method III & IV then $\exists$ a continuous and differentiable single variable function $g(y)$ such that

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$$ then $e^{\int g(y)\,dy}$ is an integrating factor of $Mdx + Ndy = 0$.

(1). Solve $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ ___ (1)

Sol:- the given D.E is of the form $Mdx + Ndy = 0$

where $M = xy^3 + y$; $N = 2x^2y^2 + 2x + 2y^4$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1 \quad ; \quad \frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\therefore \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

$\therefore$ Eqn (1) is not an E.D.E.

Let $f(x,y) = \frac{dy}{dx} = \frac{-(xy^3 + y)}{2x^2y^2 + 2x + 2y^4}$

$$f(kx,ky) = -\frac{kx\,k^3y^3 + ky}{2k^2x^2k^2y^2 + 2kx + 2k^4y^4}$$

$\therefore f(kx,ky) \neq k^0 f(x,y)$ $\therefore$ (1) is not a H.D.E.

& (1) is not in the form of $y f(x,y)dx + x g(x,y) dy = 0$ (33)

$\therefore I.F = e^{\int g(y) dy}$ check $= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$= e^{\int 1/y \, dy}$ $= \frac{1}{y}$

$= e^{\log y} = y.$ $= g(y)$

$\boxed{I.F = y}$

multiply E_q^n(1) with $I.F = y$.

$$\underbrace{y(xy^3+y)}_{M_1} dx + \underbrace{2y(x^2y^2+x+y^4)}_{N_1} dy = 0 \quad \text{———— (2)}$$

$\Rightarrow \frac{\partial M_1}{\partial y} = 4xy^3 + 2y$; $\frac{\partial N_1}{\partial x} = 4xy^3 + 2y$

$\therefore \boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$

$\therefore$ (2) is an E.D.E and the soln is

$\int M_1 dx + \int N_1 dy = c$

$\Rightarrow \boxed{3x^2y^4 + 6xy^2 + 2y^6 = 6c}$

(2). Solve $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$. (A:- $\left(y+\frac{2}{y^2}\right)x + y^2 = c$)

(3). Solve $y(xy+e^x)dx - e^x dy = 0$. (A:- $\frac{x^2}{2} + \frac{e^x}{y} = c$)

(4). $y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0$ (A:- $\frac{2}{3}\log x - \frac{1}{3}\log y - \frac{1}{3xy} = c$)

(5). $(x^2+y^2+x)dx + xy\,dy = 0$ (A:- $3x^4 + 6x^2y^2 + 4x^3 = c$)

Linear Differential Eqn :-

An Eqn of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where 'P' & 'Q' are either constant (or) functions of 'x' only is called a linear Differential Eqn (L.D.E) of 1st order interms of 'y'.

Procedure of solving :-

(i). write the Integrating factor (I.F) $= e^{\int P(x)\,dx}$

(ii). solution is $\boxed{y \times (I.F) = \int Q(x) \times (I.F)\,dx + c}$

Note :- ①. Similarly the L.D.E interms of 'x' is of the form

$$\frac{dx}{dy} + P(y)\,x = Q(y)$$

Here I.F $= e^{\int P(y)\,dy}$.

& G.S is $\boxed{x\,(I.F) = \int Q(y) \times I.F\,dy + c}$

②. $\int t e^{t}\,dt = (t-1)e^{t} + c$

③ $\int t e^{-t}\,dt = -(t+1)e^{-t} + c$.

Problems:- ①. Solve $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

Sol :- $\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{4x^2}{1+x^2}$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2} \quad \text{--- (1)}$$

$\therefore$ Eqn(1) is in the form of L.D.E

i.e, $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x) = \frac{2x}{1+x^2}$; $Q(x) = \frac{4x^2}{1+x^2}$

where $P(x) = \frac{2x}{1+x^2}$; $Q(x) = \frac{4x^2}{1+x^2}$

$\therefore I.F = e^{\int P(x)dx} = e^{\int 2x/1+x^2 dx} = e^{\log(1+x^2)} = 1+x^2$.

G.S is $y(I.F) = \int Q(x) \times (I.F)\, dx + c$

$\Rightarrow (1+x^2)y = \int \frac{4x^2}{\cancel{1+x^2}} \cancel{(1+x^2)}\, dx + c$

$\Rightarrow (1+x^2)y = \frac{4x^3}{3} + c$

(2). Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

Sol:- Divide $\cos^2 x$ on b.s.

$\Rightarrow \frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$ — (1).

$\left(\frac{dy}{dx} + Py = Q\right)$ where $P = \sec^2 x$; $Q = \tan x \sec^2 x$.

$\therefore I.F = e^{\int P(x)dx} = e^{\int \sec^2 x dx} = e^{\tan x}$

G.S is $y(I.F) = \int Q \times (I.F)\, dx + c$

$y e^{\tan x} = \int \tan x \sec^2 x \left(e^{\tan x}\right) dx + c$

$= \int t e^t dt + c$ let $\tan x = t$
$\sec^2 x dx = dt$

$= e^t(t-1) + c$

$e^{\tan x} y = e^{\tan x}(\tan x - 1) + c$

//

③. Solve $\frac{dy}{dx} + y = e^{e^x}$

Sol:- $\left(\frac{dy}{dx} + Py = Q\right)$ where $P = 1$; $Q = e^{e^x}$.

$I.F = e^{\int P(x)dx} = e^{\int 1 dx} = e^x$

G.S is $y(I.F) = \int Q \times (I.F)\, dx + c$

$ye^x = \int e^{e^x}(e^x)\, dx + c$

Put $e^x = t$

$e^x dx = dt$

$= \int e^t dt + c$

$= e^t + c$

$$\boxed{ye^x = e^{e^x} + c}$$

④. Solve $(x+y+1)\frac{dy}{dx} = 1$.

Sol:- $\frac{dy}{dx} = \frac{1}{x+y+1}$

$\Rightarrow \frac{dx}{dy} = x+y+1$

$\Rightarrow \frac{dx}{dy} - x = y+1$

$$\boxed{\frac{dx}{dy} + P(y)\,x = Q(y)},$$

$P(y) = -1$

$Q(y) = y+1$

$I.F = e^{\int P(y)dy} = e^{\int -1 dy} = e^{-y}$.

Sol is $x \times I.F = \int Q \times (I.F)\, dy + c$

$\Rightarrow xe^{-y} = \int (y+1)\, e^{-y}\, dy + c$

$$\Rightarrow x e^{-y} = \int e^{-y}(y+1)\, dy + c.$$

Applying integration by parts.

$$\left[\because \int uv\,dx = u\int v\,dx - \int\left[u'\int v\,dx\right] dx\right]$$

$$\Rightarrow x e^{-y} = (y+1)\int e^{-y}\, dy - \int\left(1 \cdot \frac{e^{-y}}{-1}\right) dy + c$$

$$= (y+1)\frac{e^{-y}}{(-1)} + \frac{e^{-y}}{(-1)} + c$$

$$= -(y+1)e^{-y} - e^{-y} + c$$

$$= -e^{-y}(y+1+1) + c$$

$$\Rightarrow \boxed{x e^{-y} = -e^{-y}(y+2) + c.}$$

5. $dr + (2r\cot\theta + \sin 2\theta)\, d\theta = 0.$

Sol:- Given $dr + (2r\cot\theta + \sin 2\theta)\, d\theta = 0.$

$$\Rightarrow \frac{dr}{d\theta} = -2r\cot\theta - \sin 2\theta$$

$$\Rightarrow \frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$$

$$\Rightarrow \frac{dr}{d\theta} + (2\cot\theta)\, r = -\sin 2\theta \quad \text{—— (1)}$$

$$\left[\because \frac{dr}{d\theta} + P(\theta)\, r = Q(\theta)\right] \quad \text{where } P(\theta) = 2\cot\theta, \; Q(\theta) = -\sin 2\theta$$

$$I.F = e^{\int P(\theta)d\theta} = e^{\int 2\cot\theta\, d\theta} = e^{2\log|\sin\theta|} = \sin^2\theta.$$

G.S is $r\,(I.F) = \int Q \times (I.F)\, d\theta + c$

$$\Rightarrow r\sin^2\theta = \int(-\sin 2\theta)(\sin^2\theta)\, d\theta + c.$$

$\Rightarrow r\sin^2\theta = -\int 2\sin\theta\cos\theta\sin^2\theta\, d\theta + c$

$= -2\int \sin^3\theta\cos\theta\, d\theta + c$ $\left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$

$= -2\frac{\sin^4\theta}{4} + c$

$\Rightarrow r\sin^2\theta + \frac{\sin^4\theta}{2} = c.$

$\Rightarrow 2r\sin^2\theta + \sin^4\theta = 2c$

$\Rightarrow \sin^2\theta\left(2r + \sin^2\theta\right) = 2c$ //

⑥. $(x+2y^3)\frac{dy}{dx} = y.$

A?- $\frac{dy}{dx} = \frac{y}{x+2y^3}.$

$\Rightarrow \frac{dx}{dy} = \frac{x+2y^3}{y}$

$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + 2y^2$

$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$

$\left[\because \frac{dx}{dy} + P(y)x = Q(y)\right]$, where $P(y) = -1/y$
$Q(y) = 2y^2.$

$I.F = e^{\int P(y)dy} = e^{\int -1/y\, dy} = e^{-[\log y]} = e^{\log y^{-1}} = 1/y.$

G.S is $x \times (I.F) = \int Q\,(I.F)\,dy + c$

$\Rightarrow \frac{x}{y} = \int \frac{2y^{\cancel{2}}}{\cancel{y}} dy + c$

$\Rightarrow \frac{x}{y} = \cancel{2}\frac{y^2}{\cancel{2}} + c \Rightarrow \boxed{x = y^3 + cy}$

7. Solve $\frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y=\frac{1}{(1+x^2)^2}$, Given that $y(0)=1$

Sol: $\left(\frac{dy}{dx}+Py=Q\right)$ where $P=\frac{2x}{1+x^2}$; $Q=1/(1+x^2)^2$

$$I.F=e^{\int P(x)dx}=e^{\int 2x/1+x^2\,dx}=e^{\log|1+x^2|}=1+x^2$$

G.S is $y\,(I.F)=\int Q\times(I.F)\,dx+C$

$$\Rightarrow y(1+x^2)=\int\frac{1}{(1+x^2)^{\cancel{2}}}\times\cancel{(1+x^2)}\,dx+C$$

$$\Rightarrow y(1+x^2)=\tan^{-1}x+C \quad \text{—— (1)}$$

Given that $y(0)=1 \Rightarrow y=1,\ x=0.$

$$\Rightarrow 1(1+0)=0+C$$

$$\Rightarrow \boxed{C=1}$$

$$(1)\Rightarrow \boxed{y(1+x^2)=\tan^{-1}x+1}$$

8. Solve $(x+1)\frac{dy}{dx}-y=e^{3x}(x+1)^2$ $\left(A:-\frac{y}{x+1}=\frac{e^{3x}}{3}+C\right)$

9. Solve $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0.$ $\left(A:-x\,e^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+C\right)$

Non-Linear (or) Bernoulli's D.E :-

An Eqn of the form $\frac{dy}{dx}+P(x)y=Q(x)y^n$ — (1) is called "Non-linear" (or) "Bernoulli's D.E", where 'P' & 'Q' are functions of 'x' alone and 'n' is a real constant

Case(1):- If $n=1$ Eq^n(1) can be written as $\frac{dy}{dx}+P(x)y=Q(x)y^1$ (40)

$$\Rightarrow \frac{dy}{dx}+[P(x)-Q(x)]\,y=0$$

$$\Rightarrow \int \frac{dy}{y}+\int [P(x)-Q(x)]\,dx=c$$

Here the variables are separable and the sol^n is

$$\int \frac{dy}{y}+\int [P(x)-Q(x)]\,dx=c$$ //.

Case(2):- If $n\neq 1$

then multiply Eq^n(1) with y^{-n}.

$$y^{-n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x) \quad \text{—— (2)}$$

$$y^{1-n}=z \quad \text{—— (3).}$$

$$(1-n)\,y^{1-n-1}\frac{dy}{dx}=\frac{dz}{dx}.$$

$$\Rightarrow (1-n)\,y^{-n}\frac{dy}{dx}=\frac{dz}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dz}{dx} \quad \text{—— (4).}$$

sub$_n$ Eq^n(4), (3) in (2)

$$\frac{1}{1-n}\frac{dz}{dx}+P(x)z=Q(x)$$

$$\rightarrow \frac{dz}{dx}+(1-n)P(x)z=Q(x)(1-n)$$

It is in the form of L.D.E and can be solved by L.D.E Procedure.

$$I.F = e^{\int P(x)\,dx}$$

sol^n is $y \times (I.F) = \int Q \times (I.F)\,dx + c$

at the end substitute $z = y^{1-n}$ then we get the required sol^n.

Note:- $^{||}$ly the Bernoulli's D.E interms of 'x' is

$$\boxed{\frac{dx}{dy} + P(y)\,x = Q(y)\,x^n}.$$

Problems

① solve the D.E $x\frac{dy}{dx} + y = x^2 y^6$.

Sol^n:- Given $x\frac{dy}{dx} + y = x^2 y^6$.

Divide the given sol^n with 'x'.

$\frac{dy}{dx} + \frac{y}{x} = x y^6$, which is in the form of — (1)

$\frac{dy}{dx} + P(x)y = Q(x)y^n$

multiply Eq^n(1) with y^{-6}

$\Rightarrow y^{-6}\frac{dy}{dx} + y^{-5}\cdot\frac{1}{x} = x$ — (2)

Let $y^{-5} = z$ — (3)

Diff$^{||}$ w.r.to 'x' on b.s

$(-5)y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dz}{dx}$ — (4).

sub$^{||}$ Eq^n(3) & (4) in Eq^n(2)

$\Rightarrow -\frac{1}{5}\frac{dz}{dx} + \frac{1}{x}\cdot z = x$

$\Rightarrow \frac{dz}{dx} - \frac{5}{x}\cdot z = -5x$ — (5)

Eq (5) is in the form of L.D.E interms of 'x'.

i.e, $\frac{dz}{dx} + P(x)z = Q(x)$

where $P(x) = -5/x$; $Q(x) = -5x$

$I.F = e^{\int P(x)dx} = e^{-\int 5/x\, dx} = e^{-5\log x} = 1/x^5$.

G.S is $z(I.F) = \int Q(x)(I.F)\,dx + C$

$\Rightarrow \frac{z}{x^5} = \int (-5x)\frac{1}{x^5}dx + C = -5\left(\frac{x^{-3}}{-3}\right) + C$

$\Rightarrow \frac{z}{x^5} = \frac{5}{3x^3} + C \Rightarrow \frac{z}{x^5} - \frac{5}{3x^3} = C$ // (or) $\frac{1}{x^5y^5} - \frac{5}{3x^3} = C$

$(\because y^{-5} = z)$

(2). Solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$.

Sol: $\frac{dy}{dx} = \frac{1}{x^2y^3 + xy}$

$\Rightarrow \frac{dx}{dy} = x^2y^3 + xy$

$\Rightarrow \frac{dx}{dy} - xy = x^2y^3$ — (1) $\left(\because \frac{dx}{dy} + P(y)\cdot x = Q(y)\,x^n\right)$

multiply (1) with x^{-2} on b.s.

$\Rightarrow x^{-2}\frac{dx}{dy} - x^{-1}y = y^3$

Put $x^{-1} = z$ — (2) $\Rightarrow (-1)x^{-2}\frac{dx}{dy} = \frac{dz}{dy} \Rightarrow x^{-2}\frac{dx}{dy} = -\frac{dz}{dy}$ — (3)

sub, (2) & (3) in (1)

$\Rightarrow -\frac{dz}{dy} - zy = y^3 \Rightarrow \frac{dz}{dy} + zy = -y^3 \quad \left(\because \frac{dz}{dy} + P(y)z = Q(y)\right)$

where $P(y) = y$; $Q(y) = -y^3$.

G.S is $z(I.F) = \int Q(y)(I.F)\,dy + c$ $\quad I.F = e^{\int P(x)dy} = e^{y^2/2}$.

$\Rightarrow z e^{y^2/2} = \int -y^3 \left(e^{y^2/2}\right) dy + c$

$\Rightarrow z e^{y^2/2} = -\int y^3 e^{y^2/2} dy + c$ $\quad \left(\because \text{put } \frac{y^2}{2} = t \Rightarrow y^2 = 2t\right.$

$\Rightarrow z e^{t} = -\int y^2 \cdot y e^{y^2/2} dy + c$ $\quad \cancel{2}y\,dy = \cancel{2}dt$

$y\,dy = dt \Big)$

$\Rightarrow z e^{t} = -\int (2t) e^{t} dt + c$

$\Rightarrow z e^{t} = -2\int t e^{t} dt \Rightarrow z e^{t} = -2 e^{t}(t-1) + c$

$\Rightarrow z e^{y^2/2} = 2 e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c.$

$\Rightarrow x^{-1} e^{y^2/2} = 2 e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c \quad (\because z = x^{-1})$

③. Solve $\frac{dy}{dx} = e^{x-y}\left(e^{x} - e^{y}\right)$.

Sol: $\frac{dy}{dx} = \frac{e^x}{e^y}\left(e^x - e^y\right) \Rightarrow e^y \frac{dy}{dx} = e^{2x} - e^x e^y \Rightarrow e^y \frac{dy}{dx} + e^x e^y = e^{2x}$ — (1)

Put $e^y = z$ — (2)

$e^y \frac{dy}{dx} = \frac{dz}{dx}$ — (3)

sub,, (2) & (3) in (1).

$\Rightarrow \frac{dz}{dx} + z e^x = e^{2x} \quad \left(\because \frac{dz}{dx} + P(x)z = Q(x)\right)$

where $P(x) = e^x$; $Q(x) = e^{2x}$

$I.F = e^{\int P(x)dx} = e^{\int e^x dx} = e^{e^x}$.

G.S is $z(I.F) = \int Q(x)(I.F)\,dx + c$

$\Rightarrow z e^{e^x} = \int e^{2x} e^{e^x} dx + c$ $\quad \left(\because \text{let } e^x = t\right.$

$e^x dx = dt \Big)$

$\Rightarrow z e^{t} = \int t\, e^{t} dt + c \Rightarrow e^y e^{e^x} = e^{t}(t-1) + c \Rightarrow e^y e^{e^x} = e^{e^x}(e^x - 1) + c$ //

H.W

⑤. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

$\Rightarrow \sec^2 y \frac{dy}{dx} + (2\tan y)x = x^3$

put $\tan y = t$

$\left(A:- \tan y\, e^{x^2} = \frac{1}{2} e^{x^2}(x^2 - 1) + c\right)$

H.W

④. Solve $\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$ (put $\tan y = z$) $\left(A:- \tan y\, e^{x^2} = \frac{1}{2} e^{x^2}(x^2 - 1) + c\right)$

⑥ solve $\frac{dy}{dx} + \frac{\tan y}{1+x} = (1+x)\,e^x \sec y.$

Sol:- Divide by "sec y"

$$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x \quad \text{—— (1)}.$$

put $\sin y = z \Rightarrow \cos y \frac{dy}{dx} = \frac{dz}{dx}.$
Diff. w.r.to 'x'

$$\Rightarrow \frac{dz}{dx} - \frac{z}{1+x} = (1+x)\,e^x$$

$$\left[\because \frac{dz}{dx} + P(x)z = Q(x)\right]$$

where $P(x) = -1/1+x$; $Q(x) = e^x(1+x)$.

$$\text{I.F} = e^{\int P(x)dx} = e^{-\int 1/1+x\, dx} = e^{-\log(1+x)} = \frac{1}{1+x}.$$

$$\text{G.S} = z \times \text{I.F} = \int Q(x)\,(\text{I.F})\,dx + C$$

$$\Rightarrow (\sin y)\frac{1}{1+x} = \int e^x \cancel{(1+x)}\, \frac{1}{\cancel{1+x}}\, dx + c$$

$$\Rightarrow \boxed{\sin y = (e^x + c)(1+x)}$$

⑦. $(1-x^2)\frac{dy}{dx} + xy = y^3 \sin^{-1}x$ $\left(\because \frac{1-x^2}{y^2} = -2\left[x\sin^{-1}x + \sqrt{1-x^2}\right] + c\right)$

⑧. $e^x \frac{dy}{dx} = 2xy^2 + y.e^x$ $\left(\frac{e^x}{y} = -x^2 + c\right)$

Sol:- $\frac{dy}{dx} = \frac{2xy^2}{e^x} + \frac{ye^x}{e^x} \Rightarrow \frac{dy}{dx} - y = \frac{2x}{e^x}y^2$

$$\left[\because \frac{dy}{dx} + P(x)y = Q(x)y^n\right]$$

Applications of an ODE :-

(i) Newton's Law of cooling

(ii). the law of Natural growth or decay

(iii). orthogonal trajectories.

(iv) Electrical circuits.

Newton's Law of cooling :- the rate of change of the temparature of the body is directly proportional to the difference of the temp of the body and that of the surrounding medium.

Let 'θ' be the temp of the body at the time 't' and 'θ_0' be the temperature of its surrounding medium (air)

By the Newton's Law of cooling we have.

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

Here 'k' is a +ve constant and Applying by the method of variable separable

$$\frac{d\theta}{(\theta-\theta_0)} = -k\,dt$$

$$\int \frac{d\theta}{(\theta-\theta_0)} = \int -k\,dt$$

$\Rightarrow \log|\theta-\theta_0| = -kt + c$. —— (1)

Let $\theta = \theta_1$ at the time $t=0$

sub., in (1)

$\Rightarrow \log|\theta_1-\theta_0| = -k(0)+c$

$\Rightarrow c = \log|\theta_1-\theta_0|$ —— (2).

sub., (2) in (1)

$\Rightarrow \log|\theta-\theta_0| = -kt + \log|\theta_1-\theta_0|$

$\Rightarrow \boxed{kt = \log|\theta_1-\theta_0| - \log|\theta-\theta_0|}$

— : Problems : —

①. the air is maintained at 30°c & the temp of the body cools down from 80°c to 60°c in 12 minutes. Find the temperature of the body after 24 minutes

Sol:- Let 'θ' be the temp of the body at time 't'.

Let 'θ_0' be the temp of the air

Given that $\theta_0 = 30°c$

Initially $\theta = 80°c$ at $t=0$

$\theta = 60°c$ at $t=12$.

we have to find 'θ' at $t=24$ minutes.

By the N-L-C,

(4.7)

W.K.T $\frac{d\theta}{dt} \propto (\theta - \theta_0)$

$\Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_0)$, where 'k' is a +ve constant

$\Rightarrow \int \frac{d\theta}{\theta - \theta_0} = -k \int dt$

$\Rightarrow \log|\theta - \theta_0| = -kt + c$

$\Rightarrow \log|\theta - 30°| = -kt + c$ — (1)

Given that $\boxed{\theta = 80°C}$ at time $\boxed{t=0}$ sub. in (1)

$\Rightarrow \log|80° - 30°| = -k(0) + c$

$\Rightarrow \boxed{c = \log|50°|}$ — (2)

sub. (2) in (1).

$\Rightarrow \log|\theta - 30°| = -kt + \log|50°|$

$\Rightarrow kt = \log|50°| - \log|\theta - 30°|$ — (3)

Given that $\boxed{\theta = 60°C}$ at $\boxed{t=12}$ sub. in (3)

$\Rightarrow 12k = \log|50°| - \log|30°|$ — (4)

$\frac{(3)}{(4)} \Rightarrow \frac{\cancel{k}t}{12\cancel{k}} = \frac{\log|50°| - \log|\theta - 30°|}{\log|50°| - \log|30°|}$

at the time $\boxed{t = 24}$, $\theta = ?$

$\Rightarrow \frac{24}{12} = \log\left[50/\theta - 30°\right] / \log(50/30)$

$\Rightarrow 2\log\left(\frac{50^\circ}{30^\circ}\right) = \log\left(\frac{50^\circ}{\theta-30^\circ}\right)$

$\Rightarrow \log\left(\frac{50^\circ}{30^\circ}\right)^2 = \log\left(\frac{50^\circ}{\theta-30^\circ}\right)$

$\Rightarrow \left(\frac{50^\circ}{30^\circ}\right)^2 = \left(\frac{50^\circ}{\theta-30^\circ}\right)$

$\Rightarrow \frac{2500}{900} = \frac{50}{\theta-30}$

$\Rightarrow \theta-30 = 18$

$\boxed{\theta = 48}$.

(2) A body kept in air with temperature $25^\circ c$ cools from $140^\circ c$ to $80^\circ c$ in 20 minutes. Find when the body cools down to $35^\circ c$.

Sol: Let 'θ' be the temp of the body at time 't'.

Let 'θ_0' be the temp of the air.

Given that $\theta_0 = 25^\circ c$.

Initially $\theta = 140^\circ c$ at $t=0$

$\theta = 80^\circ c$ at $t = 20$.

we have to find 't' at $\theta = 35^\circ c$.

By the Newton's Law of Cooling

W.K.T $\frac{d\theta}{dt} \propto (\theta - \theta_0)$

$\frac{d\theta}{dt} = -k(\theta - \theta_0)$, where '$k$' is a +ve constant

$$\Rightarrow \int \frac{d\theta}{\theta - \theta_0} = -k \int dt$$

$$\Rightarrow \log|\theta - \theta_0| = -kt + c$$

$$\Rightarrow \log|\theta - 25^\circ| = -kt + c \quad \text{——— (1)}$$

Giventhat $\theta = 140^\circ c$ at time $\boxed{t=0}$ sub. in (1)

$$(1) \Rightarrow \log|\theta - 25^\circ| = c.$$

$$\Rightarrow \log|(140^\circ - 25^\circ)| = c$$

$$\Rightarrow \boxed{c = \log|(115^\circ)|} \quad \text{——— (2)}$$

sub. (2) in (1).

$$\Rightarrow \log|\theta - 25^\circ| = -kt + \log|(115^\circ)|$$

$$\Rightarrow kt = \log|(115^\circ)| - \log|\theta - 25^\circ| \quad \text{——— (3)}$$

Giventhat $\theta = 80^\circ$ at time $\boxed{t=20}$ sub. in (3)

$$\Rightarrow 20k = \log|115^\circ| - \log|80^\circ - 25^\circ|$$

$$\Rightarrow 20k = \log|115^\circ| - \log|55^\circ| \quad \text{——— (4)}.$$

solving (3) & (4)

$$\frac{(3)}{(4)} \Rightarrow \frac{k t}{20 k} = \frac{\log|115^\circ| - \log|\theta - 25^\circ|}{\log|115^\circ| - \log|55^\circ|}$$

$\Rightarrow$ at the time $(t) = ?$ $(\theta = 35^\circ c)$

$$\Rightarrow \frac{t}{20} = \log|115^\circ| - \log(10^\circ) / \log|115^\circ| - \log|55^\circ|$$

$$\Rightarrow \frac{t}{20} = \log|115/10| \Big/ \log\left|\frac{115}{55}\right|$$

$$\Rightarrow \frac{t}{20} = \log(11.5) \Big/ \log\left(\frac{23}{11}\right) = 3.31$$

$$\Rightarrow t = 20 \times 3.31$$

$$\boxed{t = 66.2}$$

∴ the temp will be 35°C after 66.2 min.

(3). the temperature of the body drops from 100°c to 75°c in ten minutes when the surrounding air is at 20°c temperature. what will be its temperature after half an hour. when will the temparature be 25°c.

Sol:-

$\theta_0 = 20°C$

$\theta = 100°c$ at $t = 0$

$\theta = 75°c$ at $t = 10$ min.

we have to find (i). $\theta = ?$ at $t = 30$ min

(ii). $\theta = 25°$ at $t = ?$

$$\frac{(3)}{(4)} \Rightarrow \frac{\cancel{k}t}{10\cancel{k}} = \frac{\log 80° - \log(\theta - 20°)}{\log(80°) - \log(55°)}$$

(i) at $t = 30° \Rightarrow \frac{3\cancel{0}}{1\cancel{0}} = \frac{\log(80/\theta-20)}{\log(80/55)} \Rightarrow \left(\frac{80}{55}\right)^3 = \left(\frac{80}{\theta-20}\right) \Rightarrow \boxed{\theta = 45.99 \simeq 46°c}$

(ii). at $\theta = 25° \Rightarrow \frac{t}{10} = \log(80/5) \Big/ \log(80/55) \Rightarrow \boxed{t = 74}$

④. If the temperature of the air is 20°C and the temperature of the body drops from 100°C to 80°C in 10 minutes. what will be its temperature after 20 minutes. when will be the temperature 40°C. [$t = 48.2$ min & $\theta = 65°C$].

$\theta_0 = 20°C$
$\theta = 100 \to t = 0$
$\theta = 80 \to t = 10$
(i). $\theta = ? \to t = 20$
(ii). $\theta = 40°C \to t = ?$

⑤. A Pot of boiling water 100°C is removed from the fire and allowed to cool at 30°C in room temperature. Two minutes later, the temperature of the water in the pot is 90°C. what will be the temperature of the water after 5 minutes.

$\theta_0 = 30°C$; $t = 0 \to \theta = 100°C$
$t = 2 \to \theta = 90°C$
$t = 5 \to \theta = ?$ (77.46°C).

⑥. An object whose temperature is 75°C cools in an atmosphere of constant temperature 25°C at the rate of $k\theta$, 'θ' being the excess temperature of the body over that of the temperature. If after 10 minutes, the temperature of the object falls to 65°C, find its temperature after 20 minutes. Also find the time required to cool down to 55°C.

Sol:- Given that $\frac{d\theta}{dt} \propto \theta \Rightarrow \frac{d\theta}{dt} = -k\theta$.

$\Rightarrow \int \frac{1}{\theta} d\theta = -k \int dt \Rightarrow \log \theta = -kt + \dots$

(or)

$\Rightarrow \log_e \theta = -kt + \log c$

$\Rightarrow \log \theta - \log c = -kt \Rightarrow \log_e\left(\frac{\theta}{c}\right) = -kt \Rightarrow \boxed{\theta = ce^{-kt}}$ —— (1)

Initially when $\boxed{t=0} \Rightarrow \theta = 75°C - 25°C = 50°C$. (52).

$(1) \Rightarrow \log 50°C = -k(0) + C$

$\Rightarrow \boxed{C = \log 50°C}$ —— (2)

Subn (2) in (1).

$\log\theta = -kt + \log 50°$

$\Rightarrow kt = \log 50° - \log\theta$ —— (3)

Given that $\boxed{t=10}$ and $\theta = 65°C - 25°C = 40°C$

$\Rightarrow 10k = \log 50° - \log 40°C$ —— (4).

solving (3) & (4).

$$\Rightarrow \frac{\cancel{k}t}{10\cancel{k}} = \frac{\log 50° - \log\theta}{\log 50° - \log 40°}$$

$$\Rightarrow \frac{t}{10} = \frac{\log(50/\theta)}{\log(5\cancel{0}/4\cancel{0})} \quad —— (5)$$

(i). $\theta = ?$ at $t = 20min$.

$$(5) \Rightarrow \frac{2\cancel{0}}{1\cancel{0}} = \frac{\log(50/\theta)}{\log(5/4)}$$

$\Rightarrow \left(\frac{5}{4}\right)^2 = \frac{50}{\theta} \Rightarrow \frac{25}{16} = \frac{50}{\theta} \Rightarrow \boxed{\theta = 32°}$. ∴ Hence the temp after 20 minutes

$\boxed{\theta = 32°C + 25°C = 57°C}$

(ii). $t = ?$ at $\theta = 55°C - 25°C = 30°C$

$(5) \Rightarrow \frac{t}{10} = \frac{\log(50/30)}{\log(5/4)} \Rightarrow t = 10 \times \frac{0.5108}{0.2231} \Rightarrow \boxed{t = 22.89}$

//

(53)

7. A copper ball is heated to a temperature of $80^\circ C$. Then at time $t=0$ it is placed in water which is maintain at $30^\circ C$. If at $t=3$ minutes, the temperature of the ball is reduced to $50^\circ C$, find the time at which the temperature of the ball is $40^\circ C$.

Sol:- Let 'θ' be the temperature of the copper ball.

'θ_0' be the temperature of the water. i.e, $\boxed{\theta_0 = 30^\circ C}$

by N-L-C $\quad \log(\theta - \theta_0) = -kt + c$

$$\log(\theta - 30^\circ) = -kt + c \quad \text{--- (1)}$$

Given that at $\boxed{t=0}$ and $\boxed{\theta = 80^\circ C}$ subn in (1)

$$\boxed{\log(50^\circ) = c} \quad \text{--- (2)}$$

subn (2) in (1).

$$\Rightarrow \log(\theta - 30^\circ) = -kt + \log 50^\circ$$

$$\Rightarrow kt = \log 50^\circ - \log(\theta - 30^\circ) \quad \text{--- (3)}$$

Given that $\boxed{t=3}$ & $\boxed{\theta = 50^\circ C}$ subn in (3)

$$\Rightarrow 3k = \log(50^\circ) - \log(20^\circ) \quad \text{--- (4)}.$$

$$(3) \div (4) \Rightarrow \frac{kt}{3k} = \frac{\log 50^\circ - \log(\theta - 30)}{\log(50) - \log(20^\circ)} \quad \text{--- (5)}$$

We have to find $t = ?$ at $\boxed{\theta = 40^\circ C}$ subn in (5)

$$\Rightarrow \frac{t}{3} = \frac{\log(50) - \log(10)}{\log(50) - \log(20)}$$

$$\Rightarrow t = 3 \times \frac{\log(5/1)}{\log(5/2)} \Rightarrow \boxed{t = 5.27 \text{ min}}.$$

LAW OF NATURAL GROWTH OR DECAY :-

Let 'x' be the amount of a substance at time 't', the law of chemical conversion states that the rate of change of amount 'x' of a substance is directly proportional to the amount of that substance available at the time.

Natural decay: i.e., $\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = -kx$, (where '$k$' is a +ve constant $k>0$)

$$\Rightarrow \int \frac{dx}{x} = -k\int dt$$

$$\Rightarrow \log|x| = -kt + \log_e c \Rightarrow \log_e\left(\frac{x}{c}\right) = -kt$$

$$\Rightarrow \frac{x}{c} = e^{-kt} \Rightarrow \boxed{x = ce^{-kt}}$$

Natural Growth :- i.e., $\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = kx \ (k>0)$

$$\Rightarrow \int \frac{dx}{x} = \int k\,dt$$

$$\Rightarrow \log x = kt + \log c$$

$$\Rightarrow \log(x/c) = kt$$

$$\Rightarrow \frac{x}{c} = e^{kt}$$

$$\Rightarrow \boxed{x = ce^{kt}}$$

Problems :- (1). The number 'N' of bacteria in a culture grew at a rate proportional to 'N'. The value of 'N' was initially 100 and increased to 332 in one hour. What was the value of 'N' after $1\frac{1}{2}$ hours.

Sol: Given that the No. of bacteria i.e., $N=100$ at $t=0$.

and the bacteria $N = 332$ at the time $t = 1$ hour
$= 60$ min

Now, we have to find the no. of bacteria at a time $t = 1\frac{1}{2}$ hour $= 90$ min.

By the Law of Natural Growth,

$$\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = kN.$$

$$\Rightarrow \int \frac{dN}{N} = k\int dt$$

$$\Rightarrow \log N = kt + c. \quad \text{--- (1)}$$

at $\boxed{t=0}$ and $\boxed{N=100}$ sub. in (1)

$$\Rightarrow \boxed{\log 100 = c} \quad \text{--- (2)}$$

sub. (2) in (1)

$$\Rightarrow \log N = kt + \log 100.$$

$$\Rightarrow kt = -\log 100 + \log N$$

$$\Rightarrow kt = -\log 100 + \log N \quad \text{--- (3)}$$

at $\boxed{t = 60\text{min}}$ and $\boxed{N = 332}$ sub. in (3)

$$\Rightarrow 60k = \log 100 - \log 332 \quad \text{--- (4)}$$

$$(3) \div (4) \Rightarrow \frac{k t}{60 k} = \frac{-\log 100 + \log N}{-\log 100 + \log 332} \quad \text{--- (5)}$$

we have to find $N = ?$ at $\boxed{t = 90\text{min}}$ sub. in (5)

$$\Rightarrow \frac{90}{60} = \frac{-\log 100 + \log N}{\log(332/100)}$$

$$\Rightarrow \frac{3}{2} = \frac{\log(N/100)}{\log(332/100)} \Rightarrow \left(\frac{332}{100}\right)^3 = \left(\frac{N}{100}\right)^2 \Rightarrow \boxed{N = 604.9 \simeq 605}$$

2. A bacterial culture, growing Exponentially, increases from 200 to 500 grams in the period from 6 a.m to 9 a.m. How many grams will be present at noon.

Sol: Let 'N' be the number of bacteria in a culture at any time $t>0$, then

Given that $N = 200$ grams, Initially at $t=0$

$N = 500$ grams, at $t = 3$ hours (from 6 am to 9 am)

we have to find $N = ?$ at present noon time

i.e. $t = 6$ hours. (from 6 am to 12 noon)

W.K.T Law of Natural growth,

$$\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = KN$$

$$\Rightarrow \log N = Kt + C \quad \text{---} (1)$$

Given that Initially $\boxed{t=0}$ and $\boxed{N=200}$ sub, in (1)

$$\Rightarrow \boxed{\log 200 = C} \quad \text{---} (2).$$

Sub (2) in (1), we get

$$\Rightarrow \log N = Kt + \log 200$$

$$\Rightarrow Kt = \log N - \log 200 \quad \text{---} (3)$$

Given that $\boxed{t = 3 \text{ hours}}$ and $\boxed{N = 500}$ sub, in (3)

$$\Rightarrow 3K = \log 500 - \log 200 \quad \text{---} (4)$$

$$\frac{(3)}{(4)} \to \frac{t}{3} = \frac{\log(N/200)}{\log(500/200)} \Rightarrow \frac{t}{3} = \frac{\log(N/200)}{\log(5/2)} \quad \text{---} (5)$$

Put $t = 6$ in (5) $\Rightarrow (5/2)^2 = \left(\frac{N}{200}\right) \Rightarrow \frac{25}{4} = \frac{N}{200} \Rightarrow \boxed{N = 1250 \text{ grams}}$

(07)

③. The rate at which bacteria multiply is proportional to the instantaneous 'N' number present. If the original number double in 2 hrs? when it will be tripled?

Let 'N' be the No. of bacteria, let the original number be 'x'.

Sol:- Let $N = x$ at $t = 0$ (Initially)

Given $N = 2x$ at $t = 2$ hrs.

We have to find $N = 3x$ at $t = ?$

by Law of Natural growth. $\frac{dN}{dt} \propto N \Rightarrow \log N = kt + c$ — (1)

Given that Initially $\boxed{N = x}$ & $\boxed{t = 0}$

$\Rightarrow \log x = k(0) + c \Rightarrow \boxed{c = \log x}$ — (2)

Sub. (2) in (1)

$\Rightarrow \log N = kt + \log x$

$\Rightarrow kt = \log N - \log x$ — (3).

Given that $\boxed{N = 2x}$ and $\boxed{t = 2 \text{ hrs}}$

i.e., $\Rightarrow 2k = \log 2x - \log x$ — (4)

(3) ÷ (4) $\Rightarrow \frac{t}{2} = \frac{\log (N/x)}{\log (2/1)}$ — (5).

find $t = ?$ at $\boxed{N = 3x}$ sub. in (5).

$\Rightarrow \frac{t}{2} = \frac{\log (3x/x)}{\log 2}$

$\Rightarrow t = 2 \times \frac{\log 3}{\log 2}$

$\Rightarrow \boxed{t = 3.17 \text{ hrs}}$

//.

(4). If 30% of a radio active substance disappear in 10 days, how long will it take for 90% of it to disappear? (58)

Sol: Let the no. of radioactive substance is $x=100$ at $t=0$ days

After '10' days the no. of radioactive substance is $x=70$ at $t=10$ days.

($\because$ 30% disappear)

Now we need to find how long it will take to disappear 90% of radioactive substance i.e. $x=10$ at $t=?$

By the law of Natural Decay,

W.K.T $\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = -kx, \; (k>0)$

$$\Rightarrow \frac{dx}{x} = -k\,dt$$

$$\Rightarrow \log x = -kt + c \quad \text{—— (1)}$$

at $\boxed{x=100}$ & $\boxed{t=0}$ $\Rightarrow$ $\boxed{c = \log 100}$ —— (2)

Sub. (2) in (1) $\Rightarrow \log x = -kt + \log 100$

$$\Rightarrow kt = \log 100 - \log x \quad \text{—— (3)}$$

at $\boxed{x=70}$ & $\boxed{t=10}$ $\Rightarrow 10k = \log 100 - \log 70$ —— (4)

(3) $\div$ (4) $\Rightarrow \frac{t}{10} = \frac{\log(100/x)}{\log(100/70)}$ —— (5)

Put $\boxed{x=10}$ find $t=?$ $\Rightarrow \frac{t}{10} = \frac{\log(100/10)}{\log(10/7)}$

$$\Rightarrow t = \frac{10 \times \log 10}{\log(10/7)}$$

$\Rightarrow \boxed{t = 64.5}$ $\therefore$ After 64.5 days, 90% of substance will disappear.

(5). Radium decomposes at the rate of 5% of original amount in 50 years. How much will remain after 100 years.

Sol: Let 'x' be 100% of radium.

i.e, $x = 100$; Initially $t = 0$ years.

$x = 95$; (Given) $t = 50$ years. (Given)

($\because$ 5% radium is decomposing)

Now, we have to find $x = ?$ in $\boxed{t = 100 \text{ years}}$

By the law of Natural Decay, W.K.T

$$\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = -kx \Rightarrow \log x = -kt + c \quad \text{---} (1)$$

at $\boxed{t = 0}$ & $\boxed{x = 100}$

$$\boxed{c = \log 100} \quad \text{---} (2)$$

(2) in (1) $\Rightarrow \log x = -kt + \log 100$

$$\Rightarrow kt = \log 100 - \log x \quad \text{---} (3)$$

Put $\boxed{x = 95}$ & $\boxed{t = 50}$ in (3)

$$\Rightarrow 50k = \log 100 - \log 95 \quad \text{---} (4)$$

(3) ÷ (4) $\Rightarrow$

$$\frac{t}{50} = \frac{\log(100/x)}{\log(100/95)}$$

$$\Rightarrow \frac{t}{50} = \frac{\log(100/x)}{\log(20/19)} \quad \text{---} (5)$$

at $\boxed{t = 100}$, what is the value of x?

$$\Rightarrow \frac{\cancel{100}^{2}}{\cancel{50}} = \frac{\log(100/x)}{\log(20/19)}$$

$$\Rightarrow \left(\frac{20}{19}\right)^2 = \left(\frac{100}{x}\right) \Rightarrow \boxed{x = 47.5}.$$

Differential Eqⁿs of First order but not first degree:

Eg:-

① $\left(\frac{dy}{dx}\right)^2 + x^2\frac{dy}{dx} + 2y = 0$ (Second degree & 1st order)

② $\left(\frac{dy}{dx}\right)^3 + \sin x\,\frac{dy}{dx} = x^2$ (1st order & 3rd degree)

The methods to solve above Eqⁿs :-

① Solvable for 'p'

② Solvable for 'y'.

③ Solvable for 'x'

④ Clairaut's type.

Note:- The First order but not first degree Differential Eqⁿs expressed interms of 'p'.

ie, say $p = \frac{dy}{dx}$.

the above Example Eqⁿs, we can write

① ⇒ $p^2 + x^2 p + 2y = 0$.

ie, $\left(\frac{dy}{dx}\right)^2 + x^2\left(\frac{dy}{dx}\right) + 2y = 0$.

② ⇒ $p^3 + \sin x\, p = x^2$

ie, $\left(\frac{dy}{dx}\right)^3 + \sin x\left(\frac{dy}{dx}\right) = x^2$

//

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v + \sqrt{1-v^2}}{1}$$

$$\Rightarrow x\frac{dv}{dx} = \cancel{v} + \sqrt{1-v^2} - \cancel{v}$$

$$\Rightarrow \int \frac{dv}{\sqrt{1-v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \sin^{-1} v = \log x + \log c_1$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \log x c_1$$

$$\Rightarrow \boxed{\sin^{-1}(y/x) - \log x c_1 = 0}$$

$$\Rightarrow \int \frac{dv}{\sqrt{1-v^2}} = -\int \frac{dx}{x}$$

$$\Rightarrow \sin^{-1} v = -\log x + \log c_2$$

$$\Rightarrow \boxed{\sin^{-1}(y/x) - \log(c_2/x) = 0}$$

$\therefore$ solⁿ is $\left[\sin^{-1}(y/x) - \log(x c_1), \ \sin^{-1}(y/x) - \log(c_2/x)\right] = 0$

(3). Solve $\left(\frac{dy}{dx}\right)^2 - 5\left(\frac{dy}{dx}\right) + 6 = 0$.

Solⁿ:- $p^2 - 5p + 6 = 0$

$$\Rightarrow (p-3)(p-2) = 0.$$

$$\Rightarrow p - 3 = 0 \qquad\qquad p - 2 = 0$$

$$\boxed{p = 3} \qquad\qquad \boxed{p = 2}$$

$$\Rightarrow \frac{dy}{dx} = 3 \qquad\qquad \Rightarrow \frac{dy}{dx} = 2$$

$$\Rightarrow \int dy = 3\int dx \qquad\qquad \Rightarrow \int dy = 2\int dx$$

$$\Rightarrow y = 3x + c_1 \qquad\qquad \Rightarrow y = 2x + c_2$$

$$\Rightarrow \boxed{y - 3x - c_1 = 0} \qquad\qquad \Rightarrow \boxed{y - 2x - c_2 = 0}$$

$\therefore$ solⁿ is $\left(y - 3x - c_1, \ y - 2x - c_2\right) = 0$

(4). Solve the D.E $p^2 x(x-1)(x-2) = (3x^2-6x+2)^2$

Sol^n: $p^2 = \dfrac{(3x^2-6x+2)^2}{x(x-1)(x-2)}$

$p = \pm \dfrac{(3x^2-6x+2)}{\sqrt{x^3-3x^2+2x}}$

$\dfrac{dy}{dx} = \pm \dfrac{3x^2-6x+2}{\sqrt{x^3-3x^2+2x}}$

$\int dy = \pm \int \dfrac{3x^2-6x+2}{\sqrt{x^3-3x^2+2x}}\,dx.$

$y = \pm \int \dfrac{1}{\sqrt{t}}\,dt \qquad \left(\begin{array}{l} \text{put } x^3-3x^2+2x = t \\ (3x^2-6x+2)dx = dt \end{array} \right)$

$y = \pm \dfrac{t^{-1/2+1}}{-1/2+1} + c$

$y = \pm 2\sqrt{t} + c$

$(y - 2\sqrt{t} - c_1,\ y + 2\sqrt{t} - c_2) = 0.$

$\Rightarrow (y - 2\sqrt{t} - c,\ y + 2\sqrt{t} - c_2) = 0$

$\Rightarrow \left[y - 2\sqrt{x^3-3x^2+2x} - c_1,\ y + 2\sqrt{x^3-3x^2+2x} - c_2\right] = 0$

H.W

(5). $yp^2 + (x-y)p - x = 0$. [A: $(y - x - c_1,\ y^2 + x^2 - c_2) = 0$]

(6). Solve $p^2 - yp - (x^2 - xy) = 0$.

(9). $(p+y+x)(xp+y+x)(p+2x) = 0$

$(ye^x + e^x(x-1) - c_1,\ yx + \frac{x^2}{2} - c_2,\ y + x^2 - c_3) = 0$

(7). $yp^2 + (x-y)p - x = 0$.

(8). $p^2 + 2py\cot x = y^2$ $\left(\frac{c}{y} - 1 + \cos x,\ cy\sin x + 1 + \cos x\right) = 0$

(10). Solve $4xp^2 = (3x-a)^2$

Sol: $p^2 = \frac{(3x-a)^2}{4x}$

$\Rightarrow p = \pm\frac{(3x-a)}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \pm\left(\frac{3}{2}\sqrt{x} - \frac{a}{2}x^{-1/2}\right)$

$\Rightarrow \int dy = \pm\int\left(\frac{3}{2}x^{1/2} - \frac{a}{2}x^{-1/2}\right)dx$

$\Rightarrow y = \pm\left[\frac{3}{2}\frac{x^{3/2}}{(3/2)} - \frac{a}{2}\frac{x^{1/2}}{(1/2)}\right] + c$

$\Rightarrow y = \pm\left(x^{3/2} - ax^{1/2}\right) + c$

$\Rightarrow y - c = \pm x^{1/2}(x-a)$ (squaring on b.s)

$\Rightarrow \boxed{(y-c)^2 = x(x-a)^2}$

(11). $xp^2 = (x-a)^2$

Sol: $xp^2 = (x-a)^2 \Rightarrow p^2 = \frac{(x-a)^2}{x} \rightarrow p = \frac{x-a}{\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{x-a}{\sqrt{x}}$

$\Rightarrow dy = \frac{x-a}{\sqrt{x}}dx.$

$\Rightarrow \int dy = \int (x-a)x^{-1/2}dx \Rightarrow y = \int\left(x^{1/2} - ax^{-1/2}\right)dx$

$\Rightarrow y = \int x^{+1/2}dx - a\int x^{-1/2}dx$

$\Rightarrow y = \frac{x^{3/2}}{3/2} - a\frac{x^{1/2}}{1/2} + c$

$\Rightarrow \boxed{(y+c)^2 = 4x\left(\frac{x}{3} - a\right)^2}$

Solvable for 'y' :-

Singular soln :- A soln of differential Eqn doesnot consists arbitrary constants, then it is called "Singular soln."

General soln :- A soln of D.E having arbitrary constants is called "General soln."

Procedure :- Let $f(x,y,p)=0$ — (1) be the given D.E

If the Eqn (1) cannot be split up into rational and linear factors and eqn (1) is of 1st degree in 'y' then Eqn (1) can be solved for 'y'. If the degree of 'y' is '1'. Eqn (1) can be expressed in the form $y=f(x,p)$ — (2)

Diffn (2) w.r.to 'x', then we get

$$\frac{dy}{dx} = \frac{\partial}{\partial x} f(x,p) + \frac{\partial}{\partial p} f(x,p)\frac{dp}{dx} \quad \text{— (3)}$$

$\therefore$ Eqn (3) can be written as $p = G\left(x, p, \frac{dp}{dx}\right)$ — (4)

Eqn (4) is in two variables 'p' and 'x', Now solve the (4), we get the soln is $\phi(x,p,c)=0$ — (5).

Now eliminating 'p' from (1) & (5),

the general soln of Eqn (1) is $\psi[x,y,c]=0$.

1. Solve $y = a\sqrt{1+p^2}$ — (1)

Sol:- Eqn (1) cannot be solve interms of 'p' and 'x'.

∴ (1) can be solve for 'y'.

Diff. (1) w.r.t 'x' partially sides

$$\frac{dy}{dx} = a\frac{1}{2\sqrt{1+p^2}}(2p)\frac{dp}{dx}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{ap}{\sqrt{1+p^2}}\cdot\frac{dp}{dx}$$

$$\Rightarrow \cancel{p} = \frac{a}{\sqrt{1+p^2}}\cdot\cancel{p}\frac{dp}{dx}$$

$$\Rightarrow \frac{1}{a}\int dx = \int\frac{dp}{\sqrt{1+p^2}}$$

$$\Rightarrow \frac{x}{a}+c = \sinh^{-1}p.$$

$$\Rightarrow p = \sinh\left(\frac{x}{a}\right)+c$$

$$\Rightarrow \boxed{p = \sinh\left(\frac{x}{a}+c\right)} \quad \text{— (2).}$$

sub. 'p' value in (1)

G.S $y = a\sqrt{1+\sinh^2\left(\frac{x}{a}+c\right)}$

$$y = a\sqrt{1+\sinh^2\left(\frac{x}{a}+c\right)}$$

$$\boxed{y = a\cosh\left(\frac{x}{a}+c\right)}$$

//

② Solve $y + px = p^2 x^4$ — (1)

$\Rightarrow p^2x^4 - px - y = 0$

this is in the form of Q.E interms of 'p'

$p = \dfrac{x \pm \sqrt{x^2+4x^4y}}{2x^4}$ ($\because$ 'p' cannot give two rational numbers)

So, E_q^n(1) can be solved for 'y' not solved for 'p'.

Diffn (1) w.r.to 'x':

$\Rightarrow \dfrac{dy}{dx} + p(1) + x\dfrac{dp}{dx} = 4p^2x^3 + x^4\left(2p\dfrac{dp}{dx}\right)$

$\Rightarrow 2p + x\dfrac{dp}{dx} - 2px^4\dfrac{dp}{dx} - 4p^2x^3 = 0.$

$\Rightarrow x\dfrac{dp}{dx}(1-2px^3) + 2p(1-2px^3) = 0$

$\Rightarrow (1-2px^3)\left(x\dfrac{dp}{dx} + 2p\right) = 0$

$1-2px^3 = 0 \Rightarrow p = \dfrac{1}{2x^3}$ will give singular soln.

(Ignore this).

Consider $x\dfrac{dp}{dx} + 2p = 0$

$\Rightarrow 2p = -x\dfrac{dp}{dx}$

$\Rightarrow 2\displaystyle\int\frac{dx}{x} = -\int\frac{1}{p}dp$

$\Rightarrow 2\log x = -\log p + \log c$

$\Rightarrow x^2 = \dfrac{c}{p} \Rightarrow \boxed{p = \dfrac{c}{x^2}}$ will give General soln.

Sub., $P = \frac{c}{x^2}$ in (1).

$$(1) \Rightarrow y = x^4 p^2 - px.$$

$$y = x^4 \frac{c^2}{x^4} - \frac{c}{x^2} x$$

$\boxed{y = c^2 - \frac{c}{x}}$ is the General sol^n of (1)

(3). Solve the D.E $y - 2px + p^2 = 0$.

Sol:

$$y = 2px - p^2 \quad \text{——— (1)}.$$

Above Diff. w.r.to 'x'

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow p = 2p + 2x.\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow 2(x-p)\frac{dp}{dx} + P = 0 \quad \text{——— (2)}.$$

$$\Rightarrow \frac{dx}{dp} = \frac{2(x-p)}{-p}$$

$$\Rightarrow \frac{dx}{dp} + \frac{2}{p}x = 2.$$

$$\Rightarrow P(p) = \frac{2}{p}\ ;\ Q(p) = 2. \quad \left(\begin{array}{l} \because \text{L.D.E in the form of} \\ \because \frac{dx}{dp} + P(p)x = Q(p) \end{array}\right)$$

$$I.F = e^{\int P(p)dp} = e^{\int 2/p\, dp} = e^{2\log p} = p^2 \quad \left(\because \frac{dy}{dx} + P(x)y = Q(x)\right)$$

$$x(I.F) = \int Q(p)\,(I.F)\,dp + c$$

$$xp^2 = \int 2p^2\,dp + c$$

$xp^2 = \frac{2p^3}{3} + c \Rightarrow \boxed{3xp^2 - 2p^3 = c}$ is the G.S of (1)

Alternative Method :-

(2) $\Rightarrow$ Consider $2\frac{dp}{dx}(x-p)+p=0$

$$p\,dx + 2(x-p)\,dp = 0 \quad \text{--- (3)}$$

It is in the form of $M\,dx + N\,dy = 0$

$$M = p \;;\; N = 2(x-p)$$

$$\frac{\partial M}{\partial p} = 1 \;;\; \frac{\partial N}{\partial x} = 2.$$

$$\therefore \frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial x}$$ is Non-Exact D.E.

Consider $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial p}\right) = \frac{1}{p}(2-1) = 1/p. = f(p)$

$$I.F = e^{\int f(p)\,dp} = e^{\int 1/p\,dp} = e^{\log p} = p.$$

multiply (3) with $I.F = p$.

$\Rightarrow p^2 dx + 2p(x-p)\,dp = 0.$ —— (2).

$(M_1\,dx + N_1\,dp) = 0$ where $M_1 = p^2$

$N_1 = 2p(x-p)$

$$\therefore \frac{\partial M_1}{\partial p} = 2p \;;\; \frac{\partial N_1}{\partial x} = 2p.$$

$$\boxed{\therefore \frac{\partial M_1}{\partial p} = \frac{\partial N_1}{\partial x}}$$

$\therefore$ (2) is an Exact D.E.

Soln is $\underset{\text{'p' constant}}{\int M_1\,dx} + \underset{\text{don't take x-terms}}{\int N_1\,dp} = C \Rightarrow \int p^2 dx + \int 2p(x-p)\,dp = C$

$\Rightarrow p^2\int 1\,dx - 2\int p^2 dp = C$

$\Rightarrow p^2(x) - 2\frac{p^3}{3} = C$

$\Rightarrow \boxed{xp^2 - \frac{2p^3}{3} = C}$ is the G.S of (1)

(49). Solve $y = 3x + \log p$

Sol:- Given D.E is 1st order D.E,

Consider $y = 3x + \log p.$ ——— (1)

Diff w.r.to 'x'.

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx} \Rightarrow p = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\Rightarrow \frac{1}{p}\frac{dp}{dx} = p-3$$

$$\Rightarrow \int \frac{1}{p(p-3)}dp = \int dx. \quad \text{separate variables}$$

$$\Rightarrow \int \frac{-1}{3p}dp + \int \frac{1}{3(p-3)}dp = x + c$$

[$\because \frac{1}{p(p-3)} = \frac{A}{p} + \frac{B}{p-3}.$

$1 = A(p-3) + Bp$ — (2)

$p=3$ in (2) $\Rightarrow Bp = 1$

$3B = 1 \Rightarrow \boxed{B = 1/3}$

$p=0$ in (2) $\Rightarrow -3A = 1$

$\boxed{A = -1/3}$.

$\therefore \frac{1}{p(p-3)} = \frac{-1}{3p} + \frac{1}{3(p-3)}$]

$$\Rightarrow -\frac{1}{3}\log p + \frac{1}{3}\log(p-3) = x + c$$

$$\Rightarrow -\log p + \log(p-3) = 3x + c$$

$$\Rightarrow \boxed{\log\left(\frac{p-3}{p}\right) = 3x + c}$$

is the General sol of (1)

4

5. $y = p\tan p + \log\cos p$. — (1)

Sol. $\frac{dy}{dx} = p\sec^2 p \frac{dp}{dx} + \tan p \frac{dp}{dx} + \frac{1}{\cos p}(-\sin p)\frac{dp}{dx}$

$\Rightarrow \cancel{p} = \frac{dp}{dx}\left(\cancel{p}\sec^2 p + \cancel{\tan p} - \cancel{\tan p}\right)$

$\Rightarrow \int dx = \int \sec^2 p\, dp$

$\Rightarrow \boxed{x + c = \tan p}$.

6. $x^2 + xp^2 = yp$. — (1)

Sol. $y = \frac{x^2}{p} + xp$

$\frac{dy}{dx} = \frac{p(2x) - x^2\frac{dp}{dx}}{p^2} + x\frac{dp}{dx} + p$

$\cancel{p} = \frac{2x\cancel{p}}{p^{\cancel{2}}} - \frac{x^2}{p^2}\frac{dp}{dx} + x\frac{dp}{dx} + \cancel{p}$

$0 = \frac{dp}{dx}\left(x - \frac{x^2}{p^2}\right) + \frac{2x}{p}$

$-\frac{2x}{\cancel{p}} = \frac{dp}{dx}\left(\frac{p^2x - x^2}{p^{\cancel{2}}}\right)$

$-2x = \frac{dp}{dx}\left(\frac{p^2x - x^2}{p}\right)$

$\frac{dp}{dx} = \frac{-2xp}{p^2x - x^2}$

$\frac{dx}{dp} = \frac{p^2x - x^2}{-2xp}$

$\frac{dx}{dp} = \frac{-p}{2} + \frac{x}{2p} \Rightarrow \frac{dx}{dp} - \frac{x}{2p} = -\frac{p}{2}$

$$I.F = e^{\int -\frac{1}{2p}dp} = e^{-\frac{1}{2}\log p} = p^{-1/2} = \frac{1}{p^{1/2}}$$

$$x(I.F) = \int \Phi(p)(I.F)\,dp + c$$

$$\Rightarrow \frac{x}{p^{1/2}} = \int -\frac{p}{2}\frac{1}{p^{1/2}}\,dp + c$$

$$\Rightarrow \frac{x}{\sqrt{p}} = -\frac{1}{2}\int p^{1/2}\,dp + c$$

$$\Rightarrow \frac{x}{\sqrt{p}} = -\frac{1}{2}\left(\frac{p^{3/2}}{3/2}\right) + c$$

$$\Rightarrow \frac{x}{\sqrt{p}} = \frac{-p^{3/2}}{3} + c$$

$$\Rightarrow \boxed{\frac{x}{\sqrt{p}} + \frac{p^{3/2}}{3} = c}$$

7. $y = x + a\tan^{-1}p$.

Diff. w.r.to 'x':

$$\frac{dy}{dx} = 1 + a\frac{1}{1+p^2}\frac{dp}{dx}$$

$$\Rightarrow p - 1 = \left(\frac{a}{1+p^2}\right)\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx} = \frac{(p-1)(1+p^2)}{a}$$

$$\Rightarrow \frac{dx}{dp} = \frac{a}{(p-1)(1+p^2)} \quad \text{— (I)}$$

by using the partial fractions, we have to find RHS value,

$$\frac{a}{(p-1)(1+p^2)} = \frac{Ap+B}{(1+p^2)} + \frac{C}{(p-1)} \quad \text{—— (II)}$$

$$\frac{a}{(p-1)(1+p^2)} = \frac{(Ap+B)(p-1)+C(1+p^2)}{(1+p^2)(p-1)}$$

$$a = (Ap+B)(p-1) + C(1+p^2)$$

$$a = Ap^2 - Ap + Bp - B + C + Cp^2$$

$$a = p^2(A+C) + p(B-A) + (C-B)$$

compare the coefficients on b.s (1)

'p^2' coefficients $\Rightarrow A+C=0 \rightarrow (1) \Rightarrow \boxed{A=-C} \Rightarrow \boxed{A=-\frac{a}{2}}$

'p' " $\Rightarrow B-A=0 \rightarrow (2) \Rightarrow \boxed{B=A} \Rightarrow \boxed{B=-\frac{a}{2}}$

constants $\Rightarrow C-B=a \Rightarrow \cancel{C-a=a}$ $(\because \cancel{b=a})$ $\boxed{\cancel{C=2a}}$

In (2) put $\boxed{A=-C} \Rightarrow C+B=0 \rightarrow (3)$

$$2C = a$$

$$\boxed{C=\frac{a}{2}}$$

$\therefore \boxed{A=-\frac{a}{2}}$; $\boxed{B=-\frac{a}{2}}$; $\boxed{C=\frac{a}{2}}$

From (II) :- $$\frac{a}{(p-1)(1+p^2)} = \frac{Ap+B}{1+p^2} + \frac{C}{p-1}$$

$$\frac{a}{(p-1)(1+p^2)} = \frac{-\frac{a}{2}p - \frac{a}{2}}{1+p^2} + \frac{a}{2(p-1)}$$

$$\frac{a}{(p-1)(1+p^2)} = \frac{-ap}{2(1+p^2)} - \frac{a}{2(1+p^2)} + \frac{a}{2(p-1)} \quad \text{—— (III)}$$

subn (III) in (I).

$$\frac{dx}{dp} = \frac{-ap}{2(1+p^2)} - \frac{a}{2(1+p^2)} + \frac{a}{2(p-1)}$$

$$\int dx = \int\left[\frac{-ap}{2(1+p^2)} - \frac{a}{2(1+p^2)} + \frac{a}{2(p-1)}\right]dp$$

$$x = -\frac{a}{4}\log(1+p^2) - \frac{a}{2}\tan^{-1}p + \frac{a}{2}\log(p-1) + c$$

$$x = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}(1+p^2) - \tan^{-1}p\right] + c$$

⑧. Solve $x^3p^2 + x^2yp + 4 = 0$. — (1)

Sol^n:- Diff$_n$ w.r.to 'x'

$$\Rightarrow p^2(3x^2) + x^3(2p)\frac{dp}{dx} + x^2y\frac{dp}{dx} + yp(2x) + px^2\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow 3x^2p^2 + 2px^3\frac{dp}{dx} + x^2y\frac{dp}{dx} + 2pxy + px^2(p) = 0$$

$$\Rightarrow \frac{dp}{dx}(2px^3 + x^2y) = -4x^2p^2 - 2pxy$$

$$\Rightarrow \frac{dp}{dx} = \frac{-4x^2p^2 - 2pxy}{2px^3 + x^2y}$$

$$\Rightarrow \frac{dp}{dx} = \frac{-2p(2x^2p + \cancel{xy})}{x(2px^2 + \cancel{xy})}$$

$$\Rightarrow \frac{dp}{dx} = \frac{-2p}{x}$$

$$\Rightarrow \int \frac{1}{p}dp = -2\int \frac{1}{x}dx$$

$$\Rightarrow \log p = -2\log x + \log c$$

$$\Rightarrow p = (x^{-2}c)$$

$$\Rightarrow \boxed{p = \frac{c}{x^2}}$$ will give General sol^n.

Put 'p' value in (1)

$$x^3\frac{c^2}{x^4} + x^2y\frac{c}{x^2} + 4 = 0 \Rightarrow \boxed{\frac{c^2}{x} + cy + 4 = 0}$$ is the G.S of (1).

Solvable for 'x' :- Let $f(x,y,p)=0$ — (1) be the given D.E.

If the Eqⁿ (1) cannot be split up into Rational and linear factors and Eqⁿ (1) is of 1st degree in 'x', then Eqⁿ (1) can be solved for 'x'. Eqⁿ (1) can be expressed in the form as

$$x = f(y,p) \quad \text{— (2)}$$

Diffᵤ Eqⁿ (2) w.r.to 'y', we get

$$\frac{dx}{dy} = \frac{\partial}{\partial y} f(y,p) + \frac{\partial}{\partial p} f(y,p)\frac{dp}{dy} \quad \text{— (3)}$$

Eqⁿ (3) can be written as $\frac{1}{p} = G(y,p)$ — (4).

∴ Eqⁿ (4) is having two variables 'y' and 'p', it can be solved.

∴ the solⁿ of Eqⁿ (4) is $\phi(y,p,c)=0$ — (5)

Now eliminating 'p' from Eqⁿ (5) & (1), then the G.S of Eqⁿ (1) is $\psi(x,y,c)=0$ //.

Problems :- ①. Solve $xp^3 = a+bp$. — (1).

Solⁿ :- It cannot be solved for 'p' & 'y'.

It can be solved for 'x' method.

∴ The given Eqⁿ (1) can be written as.

$$x = \frac{+a+bp}{p^3}$$

$$x = \frac{a}{p^3} + \frac{b}{p^2} \quad \text{— (2)}$$

Diffᵤ (2) w.r.to 'y'.

$$\Rightarrow \frac{dx}{dy} = a(-3)p^{-4}\frac{dp}{dy} + b(-2)p^{-2-1}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \frac{-3a}{p^4}\frac{dp}{dy} - \frac{2b}{p^3}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \frac{dp}{dy}\left[\frac{-3a}{p^4} - \frac{2b}{p^3}\right]$$

$$\Rightarrow \frac{1}{\cancel{p}}\left[\frac{-3a}{p^3} - \frac{2b}{p^2}\right]\frac{dp}{dy} = \frac{1}{\cancel{p}}$$

$$\Rightarrow \frac{dp}{dy}\left(\frac{-3a}{p^3} - \frac{2b}{p^2}\right) = 1$$

$$\Rightarrow dp\left(\frac{-3a}{p^3} - \frac{2b}{p^2}\right) = dy$$

$$\Rightarrow \int \frac{-3a}{p^3}dp - \int \frac{2b}{p^2}dp = \int dy$$

$$\Rightarrow -3a\left(\frac{p^{-2}}{-2}\right) - 2b\left(\frac{p^{-1}}{-1}\right) = y + c$$

$$\Rightarrow \frac{3a}{2p^2} + \frac{2b}{p} = y + c.$$

$$y = \frac{3a}{2p^2} + \frac{2b}{p} - c \quad \text{——} \quad (3)$$

∴ It is not possible to eliminate 'p' from (1) & (3)

∴ The G.S of Eqⁿ(1) is

$$\boxed{y = \frac{3a}{2p^2} + \frac{2b}{p} - c}$$

//

(2). Solve $y^2 \log y = xyp + p^2$

Sol: $y^2 \log y = xyp + p^2$ — (1).

$\because$ 'x' is at 1^{st} degree of Eq^n(1), It can be solved for 'x'.

$$x = \frac{y^2 \log y - p^2}{yp}.$$

Divide with 'yp'

$$x = \frac{y \log y}{p} - \frac{p}{y} \quad \text{— (2)}$$

Diff^{ll} Eq^n(2) w.r.to 'y'

$$\Rightarrow \frac{dx}{dy} = \frac{p\left(y\frac{1}{y} + \log y\right) - y\log y \frac{dp}{dy}}{p^2} - \left[\frac{y\frac{dp}{dy} - p(1)}{y^2}\right]$$

$$\Rightarrow \frac{1}{p} = \frac{p(1+\log y) - y\log y \frac{dp}{dy}}{p^2} - \frac{\frac{dp}{dy}}{y} + \frac{p}{y^2}$$

$$\Rightarrow \frac{1}{p} = \frac{1+\log y}{p} - \frac{y\log y}{p^2}\frac{dp}{dy} - \frac{dp}{y\,dy} + \frac{p}{y^2}$$

$$\Rightarrow \frac{1}{p} = -\frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right) + \frac{1+\log y}{p} + \frac{p}{y^2}$$

$$\Rightarrow \frac{dp}{dy}\left[y\log y\left(\frac{1}{p^2}\right) + \frac{1}{y}\right] = \frac{p}{y^2} - \frac{1}{p} + \frac{1}{p}(1+\log y)$$

$$\Rightarrow \left(\frac{y\log y}{p^2} + \frac{1}{y}\right)\frac{dp}{dy} = \frac{p}{y^2} - \cancel{\frac{1}{p}} + \cancel{\frac{1}{p}} + \frac{1}{p}\log y$$

$$\Rightarrow \cancel{\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)}\frac{dp}{dy} = \frac{p}{y}\cancel{\left(\frac{1}{y} + \frac{y}{p^2}\log y\right)}$$

$$\Rightarrow \frac{dp}{dy} = \frac{p}{y} \Rightarrow \boxed{p = yc} \quad \text{— (3).}$$

Subn (3) in (1)

$$y^2 \log y = xy(yc) + y^2c^2$$

$$\Rightarrow y^2 \log y = xy^2 c + y^2 c^2$$

$$\Rightarrow \cancel{y^2} \log y = \cancel{y^2}(xc + c^2)$$

$$\Rightarrow \boxed{\log y = cx + c^2}$$

③. Solve. $p^3 - 4xyp + 8y^2 = 0$. — (1)

Sol:- $4xyp = p^3 + 8y^2$

$$x = \frac{p^3 + 8y^2}{4py}$$

$$\Rightarrow x = \frac{p^2}{4y} + \frac{2y}{p} \quad \text{— (2)}.$$

Diffn (2) w.r.to 'y'.

$$\Rightarrow \frac{dx}{dy} = \frac{(4y)(2p)\frac{dp}{dy} - p^2(4)}{(4y)^2} + \frac{p(2) - 2y\frac{dp}{dy}}{p^2}$$

$$\Rightarrow \frac{1}{p} = \frac{8yp\frac{dp}{dy} - 4p^2}{16y^2} + \frac{2p - 2y\frac{dp}{dy}}{p^2}$$

$$\Rightarrow \frac{1}{p} = \frac{p}{2y}\frac{dp}{dy} - \frac{p^2}{4y^2} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) + \left(\frac{2}{p} - \frac{p^2}{4y^2}\right).$$

$$\Rightarrow \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) + \frac{1}{p} - \frac{p^2}{4y^2} = 0.$$

$$\Rightarrow \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) - \frac{p}{2y}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = 0$$

$$\rightarrow \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\left(\frac{dp}{dy} - \frac{p}{2y}\right) = 0.$$

Here we omit the first term.

$$\frac{dp}{dy} = \frac{p}{2y} \Rightarrow \log p = \frac{1}{2}\log y + \log c$$

$$\Rightarrow \log p = \log y^{1/2} + \log c$$

$$\boxed{p = cy^{1/2}} \quad — (3).$$

sub.. (3) in (2)

$$\Rightarrow x = \frac{2y}{cy^{1/2}} + \frac{c^2 y}{4y}$$

$$\Rightarrow \boxed{x = \frac{2}{c}\sqrt{y} + \frac{c^2}{4}}$$ is the G.S of (1).

(4). Solve the D.E $y - 2px + ayp^2 = 0$.

Sol:- $y - 2px + ayp^2 = 0$ — (1)

$$\Rightarrow 2px = y + ayp^2$$

$$\Rightarrow x = \frac{y}{2p} + \frac{ayp^2}{2p}$$

$$\Rightarrow x = \frac{y}{2p} + \frac{ayp}{2} \Rightarrow 2x = \frac{y}{p} + ayp. — (2).$$

Diff.. w.r.to 'y'

$$\Rightarrow 2\frac{dx}{dy} = \frac{p(1) - y\frac{dp}{dy}}{p^2} + a\left(y\frac{dp}{dy} + p\right)$$

$$\rightarrow \frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} + ay\frac{dp}{dy} + ap$$

$$\Rightarrow \frac{1}{p} - ap = -\frac{dp}{dy}\left(ay + \frac{y}{p^2}\right)$$

$$\Rightarrow \frac{1}{p} - ap + \frac{dp}{dy}\left(\frac{y}{p^2} - ay\right) = 0$$

$$\Rightarrow \frac{1}{p} - ap + y\frac{dp}{dy}\left(\frac{1}{p^2} - a\right) = 0$$

$$\Rightarrow p\cancel{\left(\frac{1}{p^2} - a\right)} + y\frac{dp}{dy}\cancel{\left(\frac{1}{p^2} - a\right)} = 0 \Rightarrow \left(\frac{1}{p^2} - a\right)\left(p + y\frac{dp}{dy}\right) = 0$$

$$\Rightarrow p + y\frac{dp}{dy} = 0 \Rightarrow y\frac{dp}{dy} = -p.$$

(Here $\frac{1}{p^2} - a = 0$ will give singular soln. Ignore the soln)

$$\Rightarrow \frac{-dp}{p} = \frac{dy}{y} \Rightarrow -\log p = \log y + \log c$$

$$\Rightarrow \boxed{p^{-1} = cy} \Rightarrow \boxed{p = \frac{1}{yc}} \quad \text{--- (3)}$$

subn (3) in (2)

$$x = \frac{y(yc)}{2} + \frac{ay}{2}\left(\frac{1}{yc}\right)$$

$$\boxed{x = \frac{y^2c}{2} + \frac{a}{2c}}$$

⑤. $p = \tan\left(x - \frac{p}{1+p^2}\right)$ --- (1)

Sol: $x - \frac{p}{1+p^2} = \tan^{-1}p.$

$$\Rightarrow x = \tan^{-1}p + \frac{p}{1+p^2} \quad \text{--- (2)}$$

diffn w.r.to 'y'

$$\Rightarrow \frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - p(2p)\frac{dp}{dy}}{(1+p^2)^2}$$

$$\Rightarrow \frac{1}{p} = \frac{dp}{dy}\left[\frac{1}{1+p^2} + \frac{1}{1+p^2} - \frac{2p^2}{(1+p^2)^2}\right] \Rightarrow \frac{1}{p} = \frac{dp}{dy}\left[\frac{2}{1+p^2} - \frac{2p^2}{(1+p^2)^2}\right]$$

$$\Rightarrow \frac{1}{p} = \frac{dp}{dy}\left[\frac{2(1+p^2) - \cancel{2}p^2}{(1+p^2)^2}\right]$$

$$\Rightarrow \frac{1}{p} = \frac{dp}{dy}\left[\frac{2+\cancel{2p^2}-\cancel{2p^2}}{(1+p^2)^2}\right]$$

$$\Rightarrow \frac{1}{p} = \frac{dp}{dy}\,\frac{2}{(1+p^2)^2}$$

$$\Rightarrow \int dy = \int \frac{2p}{(1+p^2)^2}\,dp$$

$1+p^2 = t$

$2p\,dp = dt$

$$\Rightarrow \int dy = \int \frac{dt}{t^2}$$

$$y = \frac{t^{-2+1}}{-2+1} + c$$

$$y = \frac{-1}{t} + c$$

$$\boxed{y = \frac{-1}{1+p^2} + c}$$

⑥. Solve $y = 2px + p^2y$ — (1)

Sol: $2px = y - yp^2$

$$x = \frac{y - yp^2}{2p}$$

$$\Rightarrow x = \frac{y}{2p} - \frac{yp}{2} \quad \text{— (2)}$$

Diff. w.r.to 'y'.

$$\Rightarrow \frac{dx}{dy} = \frac{2p(1) - y\,2\frac{dp}{dy}}{2^2p^2} - \frac{1}{2}\left(y\frac{dp}{dy} + p\right)$$

$$\Rightarrow \frac{1}{p} = \frac{2p - 2y\frac{dp}{dy}}{4p^2} - \frac{y}{2}\frac{dp}{dy} - \frac{p}{2}.$$

$$\Rightarrow \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{dp}{dy}\frac{y}{2} - \frac{p}{2}$$

$$\Rightarrow \frac{1}{p} + \frac{p}{2} - \frac{1}{2p} + \frac{y}{2p^2}\frac{dp}{dy} + \frac{y}{2}\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{dp}{dy}\left(\frac{y}{2p^2} + \frac{y}{2}\right) + \frac{p}{2} + \frac{1}{2p} = 0.$$

$$\Rightarrow \frac{dp}{dy}\left(\frac{y}{2p^2} + \frac{y}{2}\right) + \frac{p}{2}\left(\frac{1}{1} + \frac{1}{p^2}\right) = 0$$

$$\Rightarrow \frac{y}{2}\frac{dp}{dy}\left(\frac{1}{p^2} + \frac{1}{1}\right) + \frac{p}{2}\left(1 + \frac{1}{p^2}\right) = 0$$

$$\Rightarrow \left(1 + \frac{1}{p^2}\right)\left(\frac{y}{2}\frac{dp}{dy} + \frac{p}{2}\right) = 0$$

Here we omit the 1st term

$$\frac{dp}{dy}\frac{y}{2} + \frac{p}{2} = 0$$

$$y\frac{dp}{dy} + p = 0$$

$$\Rightarrow \frac{1}{p}dp = \frac{1}{y}dy$$

$$\Rightarrow \log p = \log y + \log c$$

$$\Rightarrow \boxed{p = yc} \quad (\text{or}) \quad \boxed{\frac{p}{y} = c}$$

Put $\boxed{p = yc}$ in (2)

$$x = \frac{y}{2yc} - \frac{y(yc)}{2}$$

$$\boxed{x = \frac{1}{2c} - \frac{y^2 c}{2}}$$ is the G.S of (1)

Clairaut's form :- The D.E is of the form $\boxed{y = xp + f(p)}$ — (1)

is called "clairaut's eqn."

Diffn (1) w.r.to 'x'

$$\Rightarrow \frac{dy}{dx} = x\frac{dp}{dx} + p(1) + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \cancel{p} = x\frac{dp}{dx} + \cancel{p} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}\left[x + f'(p)\right] = 0$$

omiting the term $x+f'(p)$. ($\because$ It leads a singular soln)

$$\therefore \frac{dp}{dx} = 0$$

$$\Rightarrow \int dp = \int 0 \times dx \Rightarrow p = 0 + c \Rightarrow \boxed{p = c} \quad \text{— (2)}$$

Subn (2) in (1)

$$\Rightarrow \boxed{y = xc + f(c)}$$, is the general soln of (1).

Problems :- (1). Solve $(y - px)(p-1) = p$ — (1)

Sln:-

$$y - px = \frac{p}{p-1}$$

$$y = \frac{p}{p-1} + px. \quad \text{— (2)}$$

which is a clairaut's Eqn and the general soln of clairaut's Eqn is $\boxed{y = \frac{c}{c-1} + cx}$ ($\because$ Put $p = c$)

(2). $y = px + ap(1-p)$, where "a" is a constant.

Sln clearly it is clairaut's Eqn.

is $\boxed{y = px + f(p)}$

$\therefore$ G.S is $\boxed{y = cx + ac(1-c)}$ //

③. $y = px + (1+p^2)^{1/2}$

Sol: $y = cx + (1+c^2)^{1/2}$

④. $\sin(y - px) = p$

Sol: $y - px = \sin^{-1} p \Rightarrow y = px + \sin^{-1} p$

$\therefore \boxed{y = cx + \sin^{-1} c}$ //

⑤. Solve $p = \log(px - y)$

Sol: $e^p = px - y$

$\Rightarrow y = px - e^p$

$\therefore \boxed{y = cx - e^c}$ //

⑥. Solve the D.E $\sin px \cos y - \cos px \sin y = p$.

Sol: $[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$

$\Rightarrow \sin(px - y) = p$

$\Rightarrow px - y = \sin^{-1} p$

$\Rightarrow y = -\sin^{-1} p + px$

$\therefore$ the above eq in clairaut's form. i.e $\boxed{y = px + f(p)}$

$\therefore$ General soln is $\boxed{y = cx - \sin^{-1} c}$ $(\because p = c)$.

⑦. Solve $py = xp^2 + a$ and find singular soln.

Sol: $\Rightarrow y = \frac{xp^2}{p} + \frac{a}{p}$

$\Rightarrow y = xp + \frac{a}{p}$ —— (1).

$[\because y = px + f(p)]$

Diff. Eqn (1) w.r.to 'x'.

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \cancel{p} = \cancel{p} + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$$\Rightarrow -\frac{a}{p^2}\frac{dp}{dx} + x\frac{dp}{dx} = 0$$

$$\Rightarrow \frac{dp}{dx}\left(x - \frac{a}{p^2}\right) = 0$$

Here $x - \frac{a}{p^2} = 0$ will give singular soln.

$$x = \frac{a}{p^2}$$

$$\Rightarrow p^2 = \frac{a}{x} \Rightarrow \boxed{p = \pm\frac{\sqrt{a}}{\sqrt{x}}}.$$

Sub. 'p' value in (1).

$$(1) \Rightarrow y = px + \frac{a}{p}$$

put $p = \frac{\sqrt{a}}{\sqrt{x}} \Rightarrow y = \frac{\sqrt{a}}{\sqrt{x}}(x) + \frac{a}{\left(\frac{\sqrt{a}}{\sqrt{x}}\right)}$

$$\Rightarrow y = \sqrt{ax} + \frac{a\sqrt{x}}{\sqrt{a}}$$

$$\Rightarrow y = \sqrt{ax} + \sqrt{ax} \Rightarrow \boxed{y = 2\sqrt{ax}}$$

$$\boxed{y^2 = 4ax}$$

$\therefore \boxed{y^2 = 4ax}$ is "singular soln" of $\boxed{y = px + \frac{a}{p}}$.

$$(1) \Rightarrow y = px + \frac{a}{p}$$

put $p = -\frac{\sqrt{a}}{\sqrt{x}} \Rightarrow y = -\frac{\sqrt{a}}{\sqrt{x}}(x) + \frac{a}{\left(-\frac{\sqrt{a}}{\sqrt{x}}\right)}$

$$\Rightarrow y = -\sqrt{ax} - \frac{a\sqrt{x}}{\sqrt{a}}$$

$$\Rightarrow y = -\sqrt{ax} - \sqrt{ax}$$

$$\Rightarrow \boxed{y = -2\sqrt{ax}}$$

$$\Rightarrow \boxed{y^2 = 4ax}$$

Note :- For above Problem the G.S is $\boxed{y = cx + \frac{a}{c}}$

⑧. Find Singular soln of $y = px + p - p^2$ — (1)

It is in the form of $y = px + f(p)$.

Diff. (1) w.r.to 'x'.

$$\frac{dy}{dx} = p(1) + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\cancel{p} = \cancel{p} + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}(x + 1 - 2p) = 0.$$

Here $x + 1 - 2p = 0$ will give singular soln.

$\therefore 2p = x + 1 \Rightarrow \boxed{p = \frac{x+1}{2}}$ sub in (1).

$$y = \left(\frac{x+1}{2}\right)x + \left(\frac{x+1}{2}\right) - \left(\frac{x+1}{2}\right)^2$$

$$y = \left(\frac{x+1}{2}\right)\left[x + 1 - \left(\frac{x+1}{2}\right)\right]$$

$$y = \frac{(x+1)^2}{2}\left[1 - \frac{1}{2}\right] = \left(\frac{x+1}{2}\right)^2$$

$\therefore y = \left(\frac{x+1}{2}\right)^2$ is singular soln of (1).

$\boxed{\text{Note}}$:- The general soln of (1) is $y = cx + c - c^2$ //

⑨. Solve DE and find singular soln of $y = px + p^2$ — (1)

Sol: $\left[\because y = px + f(p)\right]$

Diff. (1) w.r.to 'x'

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$\cancel{p} = \cancel{p} + \frac{dp}{dx}(x + 2p)$$

$\rightarrow \frac{dp}{dx}(x+2p)=0$

$\therefore$ Here $x+2p=0$ will give singular soln.

$2p=-x \Rightarrow \boxed{p=-\frac{x}{2}}$ Put in (1)

from (1) $\Rightarrow y=px+p^2$

$y=-\frac{x}{2}(x)+\frac{x^2}{4}$

$y=-\frac{x^2}{2}+\frac{x^2}{4}$

$y=\frac{-\cancel{2}x^2}{\cancel{8}_4} \Rightarrow \boxed{y=-\frac{x^2}{4}}$ is singular soln of (1).

Note :- the General soln of (1) is $y=cx+c^2$

HW

Extra problem on Linear Differential Eqn's and non-linear Differential Eqn's (Bernoulli's differential Eqn's) :-

(1). $\frac{dy}{dx}+2xy=e^{-x^2}$ —— (1).

Soln: It is in the form of $\frac{dy}{dx}+P(x)\cdot y=Q(x)$ (L.D.E)

Here $P(x)=2x$; $Q(x)=e^{-x^2}$

$\therefore$ I.F $=e^{\int P(x)dx}=e^{\int 2x\,dx}=e^{x^2}$

G.S is $y\times I.F=\int Q(x)\, I.F\times dx+c$

$ye^{x^2}=\int e^{-x^2}e^{x^2}dx+c$

$\boxed{ye^{x^2}=x+c}$

(2). Solve $x\frac{dy}{dx}+y=\log x$

Sol:- Divide with 'x'

$\frac{dy}{dx}+\frac{y}{x}=\frac{\log x}{x}$ $\left[\because \frac{dy}{dx}+P(x)y=Q(x)\right]$

Here $P(x)=1/x$; $Q(x)=\log x/x$

$I.F = e^{\int P(x)dx} = e^{\int 1/x\, dx} = e^{\log x} = x.$

G.S $\Rightarrow y\times(I.F) = \int Q(x)\times I.F\, dx + c$

$\Rightarrow yx = \int \frac{\log x}{\cancel{x}} . \cancel{x}\, dx + c$

$\Rightarrow yx = \int \underset{v}{1}\ \underset{u}{\log x}\, dx + c$

$\Rightarrow yx = \log x\,(x) - \int \frac{1}{\cancel{x}}\cancel{x}\, dx + c$ $\left[\because \int uv\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)dx\right]$

$= x\log x - x + c$

$\therefore \boxed{xy = x(\log x - 1) + c}$

(3). Solve $(1-x^2)\frac{dy}{dx}+2xy = x\sqrt{1-x^2}$

Sol:- Divide with $(1-x^2)$ on b.s.

$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1-x^2} = \frac{x}{\sqrt{1-x^2}}$ $\left[\because \frac{dy}{dx}+P(x)y=Q(x)\right]$

Here $P(x) = \frac{2x}{1-x^2}$; $Q(x) = x/\sqrt{1-x^2}$

$I.F = e^{\int P(x)dx} = e^{\int 2x/1-x^2\, dx} = e^{-\int -2x/1-x^2\, dx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$

$\therefore y\times I.F = \int Q(x)\times I.F\, dx + c$

$y\times\frac{1}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}}\times\frac{1}{1-x^2}\, dx + c$

$$\Rightarrow \frac{y}{1-x^2} = \int x(1-x^2)^{-3/2} dx + c$$

$$= \int -\frac{1}{2} dt \cdot (t^{-3/2}) + c \qquad \because 1-x^2 = t, \; -2x\,dx = dt$$

$$= -\frac{1}{2}\int t^{-3/2} dt + c$$

$$= -\frac{1}{2} \frac{t^{-1/2}}{-1/2} + c$$

$$\Rightarrow \frac{y}{1-x^2} = \frac{1}{\sqrt{t}} + c.$$

$$\Rightarrow \boxed{\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c}$$

④. $\cosh x \frac{dy}{dx} + y \sinh x = 2\cosh^2 x \cdot \sinh x.$

Sol? Divide with $\cosh x$.

$$\frac{dy}{dx} + y \tanh x = 2\cosh x \sinh x$$

$$\left[\because \frac{dy}{dx} + P(x)y = Q(x)\right] \quad \text{Here } P(x) = \tanh x, \quad Q(x) = 2\cosh x \cdot \sinh x$$

$$I.F = e^{\int P(x) dx} = e^{\int \tanh x\, dx} = e^{\log|\cosh x|} = \cosh x$$

$$= e^{\int \frac{\sinh x}{\cosh x} dx}$$

$$\left[\because \int f'(x)/f(x)\, dx = \log|f(x)| + c\right]$$

Sol is $y \times I.F = \int Q(x) \times I.F\, dx + c$

$$\Rightarrow y\cosh x = \int 2\cosh^2 x \cdot \sinh x\, dx + c \qquad \left(\text{put } \cosh x = t, \; \sinh x\, dx = dt\right)$$

$$= 2\int t^2 dt + c$$

$$y\cosh x = 2t^3/3 + c = \frac{2\cosh^3 x}{3} + c$$

4

(5). $\frac{dy}{dx}+\frac{y}{x\log x}=\frac{\sin 2x}{\log x}$.

Sol:- $\left[\because \frac{dy}{dx}+P(x)y=Q(x)\right]$ Here $P(x)=1/x\log x$

$Q(x)=\sin 2x/\log x$.

$I.F=e^{\int P(x)dx}=e^{\int 1/x\log x\, dx}=e^{\int\left(\frac{\frac{1}{x}dx}{\log x}\right)}=e^{\log(\log x)}=\log x$

$\left[\because \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$

Sol^n is $y(I.F)=\int Q(x)\times I.F\, dx+c$

$\Rightarrow y\log x=\int\frac{\sin 2x}{\cancel{\log x}}\times\cancel{\log x}\,dx+c$

$\Rightarrow y\log x=-\frac{\cos 2x}{2}+c$. //

(6). Solve $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$.

$\Rightarrow \frac{dx}{dy}(1+y^2)+x-e^{\tan^{-1}y}=0$.

Sol:- $\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2}$.

$\left[\because \frac{dx}{dy}+P(y).x=Q(y)\right]$ where $P(y)=\frac{1}{1+y^2}$; $Q(xy)=\frac{e^{\tan^{-1}y}}{1+y^2}$

$IF=e^{\int P(y)dy}=e^{\int 1/1+y^2 dy}=e^{\tan^{-1}y}$

G.S is $x\,e^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}\times e^{\tan^{-1}y}dy+c$ $\left[\because x\times I.F=\int Q(y)\times I.F\,dy+c\right]$

$x\,e^{\tan^{-1}y}=\int e^t.e^t\,dt+c$ (Put $\tan^{-1}y=t$)

$\Rightarrow \frac{1}{1+y^2}dy=dt$

$\Rightarrow x\,e^{\tan^{-1}y}=\frac{e^{2t}}{2}+c$

$\Rightarrow x\,e^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+c$ //

7. $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$ —— (1).

Sol: $\left[\because \frac{dy}{dx} + P(x)y = Q(x)y^n\right]$

multiply with y^{-2} on by

$y^{-2}\frac{dy}{dx} + \left(\frac{1}{x}\right) y^{-1} = x\sin x$

put $y^{-1} = z \Rightarrow (-1)y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x\sin x.$ —— (2)

$\left[\because \frac{dz}{dx} + P(x)z = Q(x)\right]$

$I.F = e^{\int P(x)dx} = e^{\int -1/x\, dx} = 1/x.$

G.S is $z \times I.F = \int Q(x) \times I.F\, dx + C$

$\frac{z}{x} = \int -\cancel{x}\sin x \times \frac{1}{\cancel{x}} dx + C$

$\frac{z}{x} = \cos x + C$

$\Rightarrow \boxed{\frac{1}{xy} = \cos x + C}$ $\left(\because z = y^{-1} = 1/y\right)$

8. $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$

Sol: multiply with "y^2"

$y^2\frac{dy}{dx} - y^3\tan x = \sin x \cos^2 x$ —— (1).

$\longrightarrow y^2\frac{dy}{dx} - y^3\tan x = \sin x . \cos^2 x$

put $y^3 = z$

$3y^2\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow y^2\frac{dy}{dx} = \frac{1}{3}\frac{dz}{dx}.$

$\Rightarrow$ (1) becomes $\frac{1}{3}\frac{dz}{dx} - z\tan x = \sin x\cos^2 x$

$$\Rightarrow \frac{dz}{dx} - 3z\tan x = 3\sin x\cos^2 x.$$

$$\left[\because \frac{dz}{dx} + P(x)z = Q(x)\right]$$ — Here $\boxed{P(x) = -3\tan x}$

$Q(x) = 3\sin x\cos^2 x$

$$I.F = e^{\int P(x)dx} = e^{-\int 3\tan x\,dx} = e^{-3\int \frac{\sin x}{\cos x}dx} = e^{\int 3\sin x/\cos x\,dx}$$

$$= e^{3\log(\cos x)} = (\cos x)^3 = \cos^3 x.$$

$\therefore$ G.S is $z \times I.F = \int Q(x) \times I.F\,dx + c$

$$z\cos^3 x = \int 3\sin x\cos^2 x \times \cos^3 x\,dx + c$$

$$= 3\int \cos^5 x\sin x\,dx + c$$

$$= 3\int t^5(-dt) + c \qquad \left(\begin{array}{l}\because \cos x = t\\ -\sin x\,dx = dt\end{array}\right)$$

$$= -3\,t^6/6 + c$$

$$= \frac{-3\cos^6 x}{6}$$

$$z\cos^3 x = \frac{-\cos^6 x}{2}$$

$$\therefore y^3\cos^3 x = \frac{-\cos^6 x}{2} + c \qquad (\because z = y^3)$$

$$\Rightarrow (y\cos x)^3 = \frac{-\cos^6 x}{2} + c$$

9. In a chemical reaction a given substance is (94). being converted into another at a rate proportional to the amount of substance unconverted. If $(1/5)^{th}$ of the original amount has been transformed in 4-minutes, how much time will be required to transform one-half.

Sln:- Let 'x' grams be the amount of the remaining substance after 't' minutes.

$\therefore$ the differential Eqn is $\frac{dx}{dt} = -kx$, $k > 0$.

$\Rightarrow \frac{dx}{x} = -k\,dt$

$\Rightarrow \int \frac{1}{x}dx = -k\int dt$

$\Rightarrow \log x = -kt + c$ —— (1)

Let the original amount of substance be 'm' grams.

Given when $\boxed{t=0}$, $x = m \rightarrow \boxed{c = \log m}$ (2) sub. in (1)
(Initially)

$\rightarrow \log x = -kt + \log m$

$\Rightarrow kt = \log m - \log x$ —— (3),

Given, at $\boxed{t=4}$, $\boxed{x = m - \frac{m}{5} = \frac{4m}{5}}$ sub. in (3)

$\Rightarrow 4k = \log m - \log\left(\frac{4m}{5}\right)$ —— (4),

(3) $\div$ (4) $\Rightarrow \frac{kt}{4k} = \frac{\log(m/x)}{\log\left(\frac{m}{4m/5}\right)}$ —— (5)

we have to find $t = ?$ at $\boxed{x = \frac{m}{2}}$ sub in (5)

$$\Rightarrow \frac{t}{4} = \frac{\log\left(\frac{m}{m/2}\right)}{\log\left(\frac{m}{4m/5}\right)}$$

$$\Rightarrow \frac{t}{4} = \frac{\log 2}{\log(5/4)}$$

$$\Rightarrow t = 4 \times \frac{\log 2}{\log(5/4)}$$

$$\therefore \boxed{t = 12.42 \simeq 13 \text{ minutes}}$$

Assignment Questions
(ODE & MVC)
UNIT-I

Answer all the Questions

1. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after
   (i). 36 minutes.
   (ii). 24 minutes.

2. Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

3. Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$.

4. Solve the Differential Equations
   (a). $(y-x^2)dx + (x^2\cot y - x)dy = 0$
   (b). $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

5. (a). Solve $p(p-y) = x(x-y)$
   (b). Solve. $x^2 + p^2x = yp$.